"Introduction to Geostrophic Turbulence"

TRR 181 Winter School

Goslar, February 27, 2018

Marcel Oliver

From 2D turbulence to MOLES

1. The closure problem
2. 2D single layer turbulence
3. Two-layer stratified turbulence
4. LES closures

1. The closure problem

Rotating Boussinesq equations

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho^{-1} \nabla p = \boldsymbol{F} + D\boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Apply linear filter operation

$$\partial_t \overline{\boldsymbol{u}} + \overline{\boldsymbol{\nabla}} \cdot (\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + 2\boldsymbol{\Omega} \times \overline{\boldsymbol{u}} + \rho^{-1} \overline{\nabla} \overline{p} = \boldsymbol{R}(\boldsymbol{u}) + \overline{\boldsymbol{F}} + \overline{D}\overline{\boldsymbol{u}}$$
$$\overline{\nabla} \cdot \overline{\boldsymbol{u}} = 0$$

with *eddy source term*

$$\boldsymbol{R}(\boldsymbol{u}) = \overline{\boldsymbol{\nabla}} \cdot (\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) - \overline{\boldsymbol{\nabla} \cdot (\boldsymbol{u} \otimes \boldsymbol{u})} + \overline{D\boldsymbol{u}} - \overline{D}\overline{\boldsymbol{u}}$$

2. Barotropic vorticity equation

$$\partial_t \zeta + \boldsymbol{\nabla}^\perp \psi \cdot \boldsymbol{\nabla} \zeta = F + D_\mathrm{i} \zeta + D_\mathrm{u} \zeta$$
$$\zeta = \Delta \psi$$

Inviscid undamped equation conserves energy and enstropy

$$E = \int |\boldsymbol{\nabla} \psi|^2 \, \mathrm{d}x \qquad \text{and} \qquad Z = \int |\zeta|^2 \, \mathrm{d}x$$

Fourier representation

$$\partial_t \zeta_{\boldsymbol{k}} + \frac{1}{2\pi} \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} \frac{\boldsymbol{p}^\perp \cdot \boldsymbol{q}}{p^2} \, \zeta_{\boldsymbol{p}} \, \zeta_{\boldsymbol{q}} = D_\mathrm{i}(\boldsymbol{k}) \, \zeta_{\boldsymbol{k}} + D_\mathrm{u}(\boldsymbol{k}) \, \zeta_{\boldsymbol{k}} + F_{\boldsymbol{k}}$$

Rate of energy transfer for mode $\boldsymbol{k}$:

$$\partial_t E_{\boldsymbol{k}} = \sum_{\{\boldsymbol{p},\boldsymbol{q}\}:\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} T(\boldsymbol{k}|\boldsymbol{p}\boldsymbol{q}) + 2\, D_\mathrm{i}(\boldsymbol{k}) \, E_{\boldsymbol{k}} + 2\, D_\mathrm{u}(\boldsymbol{k}) \, E_{\boldsymbol{k}} + P_{\boldsymbol{k}}$$

where

$$T(\boldsymbol{k}|\boldsymbol{p}\boldsymbol{q}) = \frac{1}{2\pi} \, \boldsymbol{p}^\perp \cdot \boldsymbol{q} \, (q^2 - p^2) \, \mathrm{Re}[\psi_{\boldsymbol{k}} \, \psi_{\boldsymbol{p}} \, \psi_{\boldsymbol{q}}]$$

3. Resonant triad interactions (Fjørtoft, 1953)

Resonant triad: $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$

$$T(\boldsymbol{p}|\boldsymbol{k}\boldsymbol{q}) = -\frac{q^2 - k^2}{q^2 - p^2} T(\boldsymbol{k}|\boldsymbol{p}\boldsymbol{q})$$

and

$$T(\boldsymbol{q}|\boldsymbol{k}\boldsymbol{p}) = -\frac{k^2 - p^2}{q^2 - p^2} T(\boldsymbol{k}|\boldsymbol{p}\boldsymbol{q})$$

Energy transfer constraints

Suppose

$$p < k < q$$

If center mode $\boldsymbol{k}$ loses energy, modes $\boldsymbol{p}$ and $\boldsymbol{q}$ gain energy, and vice versa

4. Folklore

> Energy goes upscale
>
> Enstrophy goes downscale

Questions

- Can we prove it?
- Under which conditions?
- And in the real ocean/atmosphere?

4.1. “Proof” 1: Transient dynamics of spectral peak

Mean of spectral distribution (“Energy wavenumber”)

$$k_\mathrm{e} = \frac{1}{E} \sum_k k\, E(k)$$

Variance of spectral distribution

$$I = \sum_k (k - k_\mathrm{e})^2\, E(k) = Z - k_\mathrm{e}^2\, E$$

Conservation of energy and enstrophy implies:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -E\, \frac{\mathrm{d}k_\mathrm{e}^2}{\mathrm{d}t}$$

As energy variance increases, the energy mean goes upscale

4.2. "Proof" 2: Infinite ranges

- Forcing scale k_{f}
- Infrared dissipation scale k_{i}
- Ultraviolet dissipation scale k_{u}

$$\boxed{k_{\mathrm{i}} \ll k_{\mathrm{f}} \ll k_{\mathrm{u}}}$$

Forcing/dissipation balance

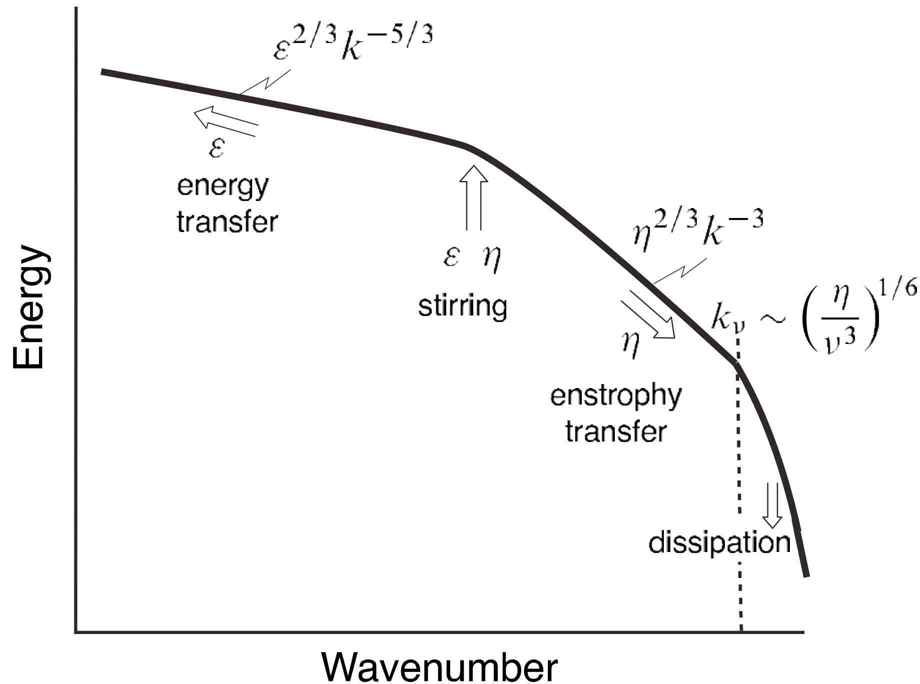
$$\varepsilon = \varepsilon_{\mathrm{i}} + \varepsilon_{\mathrm{u}}$$
$$\eta = k_{\mathrm{f}}^2\, \varepsilon = \eta_{\mathrm{i}} + \eta_{\mathrm{u}}$$

Since $\eta_{\mathrm{i}} \le k_{\mathrm{i}}^2\, \varepsilon_{\mathrm{i}}$, we estimate

$$\eta \ge \eta_{\mathrm{u}} = k_{\mathrm{f}}^2\, \varepsilon - \eta_{\mathrm{i}} \ge k_{\mathrm{f}}^2\, \varepsilon - k_{\mathrm{i}}^2\, \varepsilon_{\mathrm{i}} \ge (k_{\mathrm{f}}^2 - k_{\mathrm{i}}^2)\, \varepsilon = (1 - k_{\mathrm{i}}^2/k_{\mathrm{f}}^2)\, \eta$$

Thus, $\eta_{\mathrm{u}} \to \eta$ in the limit $k_{\mathrm{i}}/k_{\mathrm{f}} \to 0$. Similarly, $\varepsilon_{\mathrm{i}} \to \varepsilon$ when $k_{\mathrm{f}}/k_{\mathrm{u}} \to 0$.

5. Cascade picture for two-dimensional turbulence



Graphics from Vallis (2005)

6. Kolmogorov: Energy cascade

$$E_{[\kappa,2\kappa]} = \int_{\kappa}^{2\kappa} S(k)\,\mathrm{d}k \equiv g(\varepsilon,\kappa)$$

Rescaling time $t = Tt'$ and space $x = Lx'$, we demand

$$E'_{\kappa',2\kappa'} = g(\varepsilon',\kappa') \equiv g(1,1)$$

so that $\varepsilon = L^2\,T^{-3}\,\varepsilon'$, $\kappa = L^{-1}\,\kappa'$, and therefore

$$E_{[\kappa,2\kappa]} = \frac{L^2}{T^2}\,E'_{\kappa',2\kappa'} = g(1,1)\,\frac{\varepsilon^{2/3}}{\kappa^{2/3}}$$

Differentiating, we find

$$S(\kappa) - S(2\kappa) = \frac{2\,g(1,1)}{3}\,\frac{\varepsilon^{2/3}}{\kappa^{5/3}}$$

so that

$$S(\kappa) = S(2^{m+1}\kappa) + \frac{2\,g(1,1)}{3}\left(1 + \frac{1}{2^{5/3}} + \frac{1}{2^{10/3}} + \dots\right)\frac{\varepsilon^{2/3}}{\kappa^{5/3}} \quad \Longrightarrow \quad \boxed{S(k) \sim c\,\frac{\varepsilon^{2/3}}{k^{5/3}}}$$

7. Kraichnan–Bachelor–Leith: Enstrophy cascade

$$E_{[\kappa,2\kappa]} = \int_{\kappa}^{2\kappa} S(k)\,\mathrm{d}k \equiv g(\eta,\kappa)$$

Rescaling time $t = Tt'$ and space $x = Lx'$, we demand

$$E'_{\kappa',2\kappa'} = g(\eta',\kappa') \equiv g(1,1)$$

so that $\eta = T^{-3}\,\eta'$, $\kappa = L^{-1}\,\kappa'$, and therefore

$$E_{[\kappa,2\kappa]} = \frac{L^2}{T^2}\,E'_{\kappa',2\kappa'} = g(1,1)\,\frac{\eta^{2/3}}{\kappa^2} \qquad \Longrightarrow \qquad \boxed{S(k) \sim c\,\frac{\eta^{2/3}}{k^3}}$$

Dissipation length scale

$$\eta = \int_0^{k_\nu} (\nu k^2)\,(k^2)\,S(k)\,\mathrm{d}k = c\,\nu\,\eta^{2/3} \int_0^{k_\nu} k\,\mathrm{d}k = \tilde{c}\,\nu\,\eta^{2/3}\,k_\nu^2$$

so that

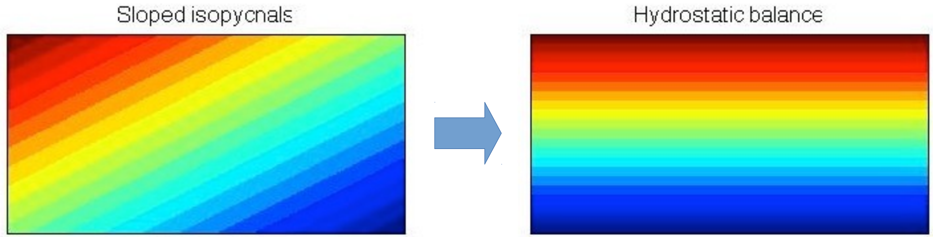
$$\boxed{k_\nu = C\,\frac{\eta^{1/6}}{\nu^{1/2}}}$$

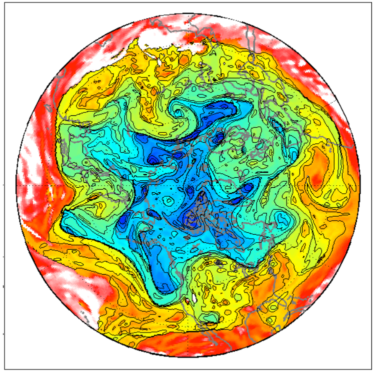
8. Limitations of the cascade picture

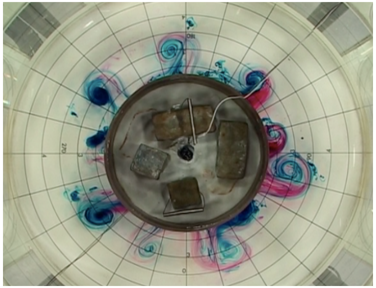
In reality

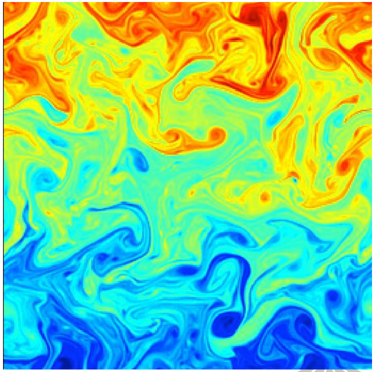
- Forcing is not spectrally localized (baroclinic instability!)
- Dissipation is not spectrally localized
- Forcing is too close to the grid scale
- Non-local triads can be significant

9. Baroclinic instability









Graphics from Sheane Keating,

https://www.climatescience.org.au/sites/default/files/Baroclinic%20Instability%20Keating%20.pdf

10. Two-layer QG as a model for ocean turbulence

Basic state: Uniform vertical shear

$$\psi_1^{\text{equil}} = -Uy \qquad \text{and} \qquad \psi_2^{\text{equil}} = 0$$

Eddy stream functions

$$\psi_1 = -yU + \psi_1^{\text{eddy}} \qquad \text{and} \qquad \psi_2 = \psi_2^{\text{eddy}}$$

Eddy barotropic and baroclinic stream functions

$$\psi = \frac{\psi_1^{\text{eddy}} + \psi_2^{\text{eddy}}}{2} \qquad \text{and} \qquad \tau = \frac{\psi_1^{\text{eddy}} - \psi_2^{\text{eddy}}}{2}$$

and associated PVs

$$q = \Delta\psi \qquad \text{and} \qquad \omega = \Delta\tau - k_{\text{d}}^2\,\tau$$

10.0.1. Two-layer QG, ctd.

$$\partial_t q + [\psi, q] + [\tau, \omega] + \frac{U}{2}\,\partial_x(q + \Delta\tau) = \frac{1}{2}\,D_\mathrm{i}(\psi - \tau) + D_\mathrm{u}\psi$$
$$\partial_t \omega + [\psi, \omega] + [\tau, q] + \frac{U}{2}\,\partial_x(\omega + q + k_\mathrm{d}^2\,\psi) = -\frac{1}{2}\,D_\mathrm{i}(\psi - \tau) + D_\mathrm{u}\tau + \kappa\tau$$

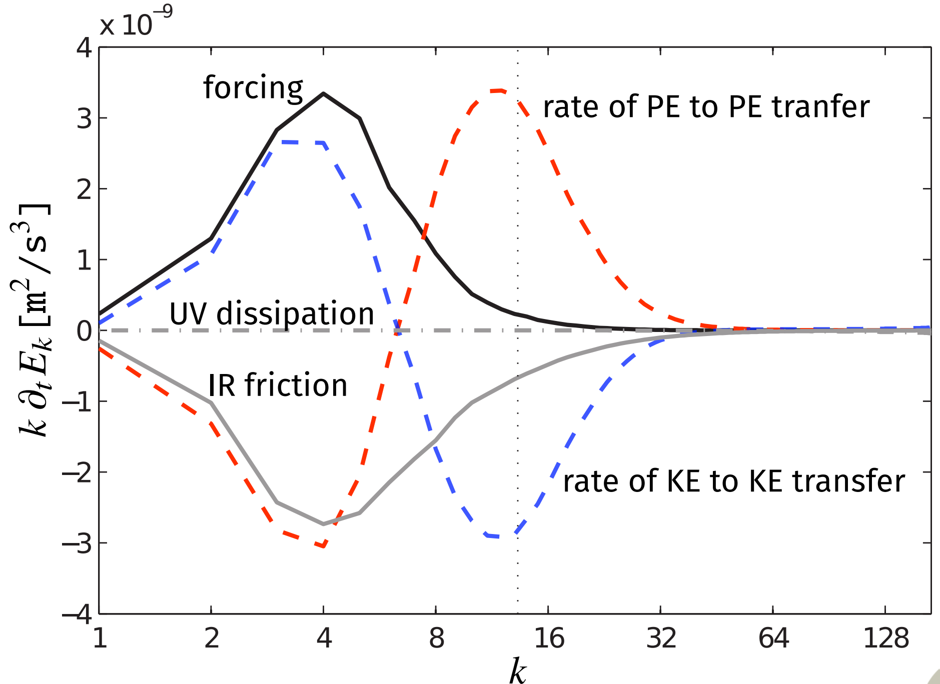
Energies (by mode)

$$E^\psi = -\frac{1}{2}\sum_{\boldsymbol{k}} k^2\,|\psi_{\boldsymbol{k}}|^2 \qquad \text{and} \qquad E^\tau = \frac{1}{2}\sum_{\boldsymbol{k}}(k^2 + k_\mathrm{d}^2)\,|\tau_{\boldsymbol{k}}|^2$$

Energies (kinetic and potential)

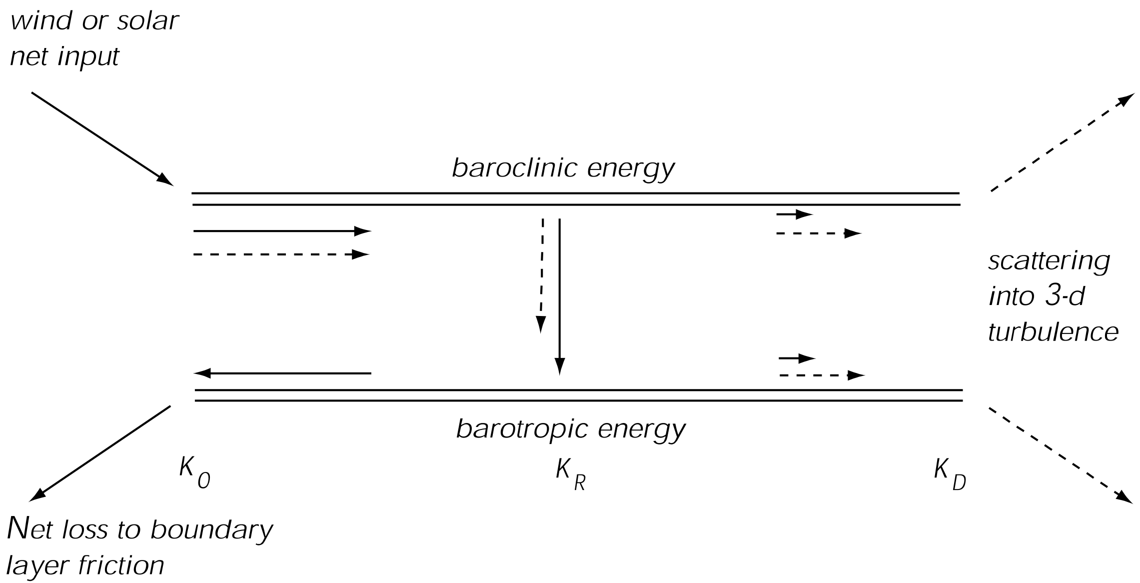
$$E^K = -\frac{1}{2}\sum_{\boldsymbol{k}} k^2\,(|\psi_{\boldsymbol{k}}|^2 + |\tau_{\boldsymbol{k}}|^2) \qquad \text{and} \qquad E^P = \frac{1}{2}\sum_{\boldsymbol{k}} k_\mathrm{d}^2\,|\tau_{\boldsymbol{k}}|^2$$

10.0.2. Two-layer QG, numerical energy transfer rates



Graphics adapted from Jansen/Held (2014)

10.1. Energy transfers in two-layer geostrophic turbulence



Graphics from Vallis (2005)

10.2. How does this apply to the ocean?

- Upper ocean: $N \approx 0.0025\mathrm{s}^{-1}$, $H \approx 4\,\mathrm{km}$, $f_0 \approx 10^{-4}\mathrm{s}^{-1}$
- Baroclinic Rossby radius: $L_\mathrm{d} \approx 100\,\mathrm{km}$
- Most unstable scale: approx. $4\,L_\mathrm{d} \approx 400\,\mathrm{km}$
- Locally much smaller values possible
- At wave numbers larger than k_d, ageostrophic effects come in
- At wave numbers around k_d, eddy viscocity comes in
- There are also surface modes, possibly higher baroclinic modes

11. Viscous parameterization (Leith)

Idea 1: Determine "eddy viscocity" ν_* such that a given dissipation wavenumber, *within the inertial range*, maintains the rate of enstropy dissipation:

$$k_* = C\,\frac{\eta^{1/6}}{\nu_*^{1/2}}$$

On the other hand, from barotropic vorticity equation,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\zeta^2 + \frac{1}{2}\,\boldsymbol{\nabla}\cdot(\boldsymbol{u}\,\zeta^2) &= \nu_*\,\zeta\,\Delta\zeta && + \text{IR friction} + \text{forcing}\\ &= -\nu_*\,|\boldsymbol{\nabla}\zeta|^2 + \nu_*\,\boldsymbol{\nabla}\cdot(\zeta\,\boldsymbol{\nabla}\zeta) && + \text{IR friction} + \text{forcing}\end{aligned}$$

Thus, area-mean-rate of enstropy dissipation $\eta = \nu_*\,\langle|\boldsymbol{\nabla}\zeta|^2\rangle$
Idea 2: Swap local and global values

$$\nu_* = \left(\frac{C}{k_*}\right)^3 \langle|\boldsymbol{\nabla}\zeta|^2\rangle^{\frac{1}{2}} = \left(\frac{C}{k_*}\right)^3 |\boldsymbol{\nabla}\zeta|$$

$\to$ MOLES (Fox–Kemper and Menemenlis, 2008)

12. Outlook

- Discrete, small-stencil operators (WIP)
- Backscatter (Stephan + Anton’s talks later)
- Can we get off the cascade picture? (Non-local triads!)
- Stochastic closures (M2 and others)
- Wave-turbulence interaction (IDEMIX!)