Models of Weather and Climate

Summer Semester 2025, Exercise 2

Due Thursday, May 29, 2025

In class, we have derived a linear response formula for a time-independent perturbation of a dynamical system with a physical measure. Namely, consider the perturbed system

$$\dot{x} = f(x) + \delta f$$

and suppose that the unperturbed system $\dot{x} = f(x)$ has a physical measure ρ. Provided that δf is independent of both t and x, the perturbed expectation of the state variable x with respect to the physical measure is given by

$$\delta\rho(x) = \int_0^\infty \int M(t,x)\,\mathrm{d}\rho(x)\,\mathrm{d}t\,\delta f\,,$$

where $M(t;x_0)$ is the fundamental matrix of the linear system

$$\delta\dot{x} = Df(x(t))\,\delta x$$

where $x(t)$ solves $\dot{x} = f(x)$ with $x(0) = x_0$.

1. Write a code that computes

$$\bar{M}(t) = \int M(t,x)\,\mathrm{d}\rho(x)$$

where the values of x are sampled from a long-time integration of the Lorenz system (as a substitute for sampling from the physical measure) for different values of t.

2. Write a code that computes

$$\delta\rho(x) \approx \int_0^T \bar{M}(t)\,\delta f\,\mathrm{d}t\,.$$

3. In Exercise 1, Problem 3 you have already set up a code where you can test, using the Lorenz system, the response of $\rho(x)$ to different δf by direct simulation. Adapt the code to directly compute $\delta\rho(x)$ for selected δf and compare your result with that in Problem 2.