Models of Weather and Climate

Summer Semester 2025, Exercise 1

Due Thursday, May 8, 2025

1. (From Kaper/Engler, Chapter 7, Exercises 2, 3.)

   (a) Let $A \colon t \mapsto A(t)$ be a differentiable function defined on a neighborhood of $t = 0$ whose values are real $n \times n$ matrices, and suppose that $A(0) = I$, the $n \times n$ identity matrix. Prove that
$$\Bigl(\frac{\mathrm{d}}{\mathrm{d}t} \det A(t)\Bigr)(0) = \operatorname{Tr} \frac{\mathrm{d}A}{\mathrm{d}t}(0) \,.$$

   (b) Let ϕ_t be the flow of the initial value problem
$$\dot{x} = f(x) \,,$$
i.e., $x(t) = \phi_t(x_0)$ is a solution to the differential equation with $x(0) = x_0$. Assume, for simplicity, that $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and that the flow is defined on all of $\mathbb{R}^n$ and for all positive times. Prove that, for all $x \in \mathbb{R}^n$,
$$\Bigl(\frac{\mathrm{d}}{\mathrm{d}t} \det D\phi_t\Bigr)(0) = \nabla \cdot f \,.$$

   (c) Let $U \in \mathbb{R}^3$ be a measurable set. Apply the result from (b) to the Lorenz equations
$$\begin{aligned} \dot{x} &= -\sigma\,(x - y) \,, \\ \dot{y} &= \rho\, x - y - x\, z \,, \\ \dot{z} &= -\beta\, z + x\, y \end{aligned}$$
to show that, for all t,
$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Vol}(\phi_t(U)) = -(\sigma + \beta + 1) \operatorname{Vol}(\phi_t(U)) \,.$$
Use this result to explain why the attractor $\mathcal{A}$ for the Lorenz equations cannot have a subset of positive volume.

2. Recall from class that the definition of the (leading) Lyapunov exponent λ implies the following. Suppose $x(t)$ and $y(t)$ are two solutions to the differential equation $\dot{x} = f(x)$. Write $\xi(t) \equiv x(t) - y(t)$ to denote their difference. Then
$$\|\xi(t)\| \approx \mathrm{e}^{\lambda t}\, \|\xi(0)\|$$
so long as the initial perturbation $\xi(0)$ is small and t is not too large.

(a) Use this property to estimate the leading Lyapunov exponent for the Lorenz system with standard parameters experimentally.

(b) Suppose you are able to specify the initial condition with an error of 10^{-3}. For how long do you expect you can predict the evolution of the system until the error in the solution exceeds 10% of its maximal value on the attractor? All errors are measured in the Euclidean norm in $\mathbb{R}^3$.

3. Suppose you add the forcing term $\gamma \sin t$ to the x-component equation and $\gamma \cos t$ to the y-component equation. Determine experimentally whether $\langle x \rangle$, $\langle z \rangle$, and/or $\langle x^2 \rangle$ respond linearly to small changes in γ. How large can γ become before the response of the system is not smooth in γ?