Exercise 7 – Solutions

1.(a) A complete binary tree of height h where all levels are maximally filled has $2^{h+1}-1$ nodes.

If array indexing starts at 0, this means that the left-most node at depth d has index

$$i_d = 2^d - 1$$

in the array-based representation of the tree. Any node at level d therefore has index

$$i(p) = i_d + j \qquad , \; j = 0, \ldots, 2^d - 1$$

Thus, its left child has index

$$i(\ell) = i_{d+1} + 2j = 2^{d+1} - 1 + 2j = 2(2^d + j) - 1 = 2(i_d + 1 + j) - 1 = 2i(p) + 1$$

(b) As the right child is directly next to the left child in array order,

$$i(r) = 2i(p) + 2$$

(c) If c is left child of p, $i(p) = \frac{i(c)-1}{2}$. As $i(c)$ is odd, $\frac{i(c)-1}{2} = \left\lfloor \frac{i(c)-1}{2} \right\rfloor$

if c is right child of p, $i(p) = \frac{i(c)-2}{2}$. As $i(c)$ is even, $\frac{i(c)-2}{2} = \left\lfloor \frac{i(c)-2}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{i(c)-1}{2} \right\rfloor$

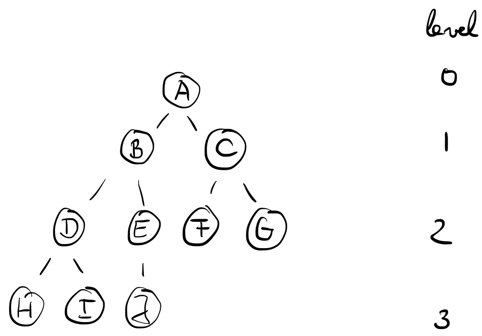
2. In a pre-order traversal, a parent has lower rank than any of its children. So heap order is always satisfied. Thus, the tree is a heap iff it is complete.

3. (1, D), (3, J), (4, B), (5, A), (2, H), (6, L)

4. Any position at level 1 or 2.

5. Level h (height of tree)

6. A B C D E F G H I J



(2, 5, 16, 4, 10, 23, 39, 18, 26, 15)

(2, 5, 16, 26, 15, 23, 39, 18, 4, 10) ← restore heap order at level 2

(2, 26, 39, 18, 15, 23, 16, 5, 4, 10) ← restore heap order at level 1

(39, 26, 23, 18, 15, 2, 16, 5, 4, 10) ← restore heap order at level 0

Now the max-heap is prepared – Phase 1 is finished.

$(26, 18, 23, 10, 15, 2, 16, 5, 4 \mid 39)$

$(23, 18, 16, 10, 15, 2, 4, 5 \mid 26, 39)$

$(18, 15, 16, 10, 5, 2, 4 \mid 23, 26, 39)$

$(16, 15, 4, 10, 5, 2 \mid 18, 23, 26, 39)$

$(15, 10, 4, 2, 5 \mid 16, 18, 23, 26, 39)$

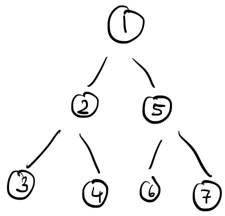
$(10, 5, 4, 2 \mid 15, 16, 18, 23, 26, 39)$

$(5, 2, 4 \mid 10, 15, 16, 18, 23, 26, 39)$

$(4, 2 \mid 5, 10, 15, 16, 18, 23, 26, 39)$

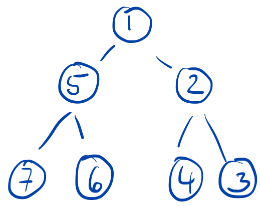
$(2, 4, 5, 10, 15, 16, 18, 23, 26, 39)$

7. pre-order traversal, increasing order:



(decreasing order not possible)

post-order traversal, decreasing order:



(increasing order not possible)

in-order traversal: never possible (will always violate heap order)

8.
$$\sum_{i=1}^{n} \log i \geq \int_1^n \log x \, dx = x \log x \Big|_1^n - \int_1^n x \frac{1}{x \ln 2} \, dx$$

$$= n \log n - \frac{n-1}{\ln 2} = \Omega(n \log n)$$