Algorithms and Data Structures

Summer Semester 2024

For discussion on Wednesday, April 24, 2022

1. (GTG Exercise R-3.2) The number of operations executed by algorithms A and B is $8n \log n$ and $2n^2$, respectively. Determine n_0 such that A is better than B for $n \geq n_0$.
2. (GTG Exercise R-3.5) Explain why the plot of the function n^γ is a straight line with slope γ on a log-log scale.
3. (GTG Exercise R-3.6) What is the sum of all the even numbers from 0 to $2n$, for any positive integer n?
4. (GTG Exercise R-3.7) Show that the following two statements are equivalent:
   (a) The running time of algorithm A is always $O(f(n))$.
   (b) In the worst case, the running time of algorithm A is $O(f(n))$.
5. (GTG Exercise R-3.11) Show that if $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n) + e(n)$ is $O(f(n) + g(n))$.
6. (GTG Exercise R-3.12) Show that if $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n) - e(n)$ is not necessarily $O(f(n) - g(n))$.

1. $8n \log n \leq 2n^2 \iff \log n \leq \frac{1}{4} n$

It is not possible to solve for n in closed form, so have to try.

E.g. $n = 16 = 2^4$: $\log n = 4$; $\frac{1}{4} n = 4$. Since log is concave down, we can take $n_0 = 16$ to conclude $8n \log n \leq 2n^2$ for $n \geq 16$.

2. $t = n^\gamma \Rightarrow \log t = \gamma \log n$. So its graph in a $\log n - \log t$ coordinate system is a straight line through the origin with slope γ.

3. $0 + 2 + 4 + \ldots + 2n = 2 \sum_{i=0}^{n} n = 2 \frac{n(n+1)}{2} = n(n+1)$

4. Let $g(I)$ denote the run-time of algorithm A given input I. Then the worst-case run-time is defined

$$w(n) = \max\{g(I) : \text{size}(I) = n\} \qquad (*)$$

Let I_n be a sequence of inputs where the maximum in $(*)$ is attained for every n.

(a) $\Rightarrow$ (b): By (a), $g(I_n) = O(f(n))$

Since $g(I_n) = w(n)$, the claim is obvious.

(b) $\Rightarrow$ (a): By (b), $w(n) = O(f(n))$

$\Rightarrow \exists\, c, n_0$ s.t. $w(n) \le c\, f(n) \quad \forall\, n \ge n_0$

Now let J_n be any other sequence of inputs where $\text{size}(J_n) = n$.

Then

$$g(J_n) \le g(I_n) = w(n) \le c\, f(n) \quad \forall\, n \ge n_0$$

$$\Rightarrow \quad g(J_n) = O(f(n)) \qquad \square$$

5. $d(n) = O(f(n)) \Rightarrow \exists\, \hat{c}, \hat{n}_0$ s.t. $d(n) \le \hat{c}\, f(n) \quad \forall\, n \ge \hat{n}_0$

$e(n) = O(g(n)) \Rightarrow \exists\, \tilde{c}, \tilde{n}_0$ s.t. $e(n) \le \tilde{c}\, g(n) \quad \forall\, n \ge \tilde{n}_0$

Now choose $c = \max\{\hat{c}, \tilde{c}\}$ and $n_0 = \max\{\hat{n}_0, \tilde{n}_0\}$.

$$\Rightarrow \quad d(n) + e(n) \le \hat{c}\, f(n) + \tilde{c}\, g(n) \le c\,(f(n) + g(n)) \quad \forall\, n \ge n_0$$

$$\Rightarrow \quad d(n) + e(n) = O(f(n) + g(n))$$

6. Counter-example: $d(n) = 2n = O(n)$; $e(n) = n = O(n)$

But: $d(n) - e(n) = n \ne O(n-n) = 0$