Algorithms and Data Structures

Mock Exam

July 15, 2025

1. Consider the function `f(x,n)` where `x` is assumed to be an object of a number type and `n` is assumed to be a nonnegative integer.

```
def f(x, n):
    if n == 0:
        return 1
    elif n%2 == 0:
        return f(x, n//2) * f(x, n//2)
    else:
        return x*f(x, n-1)
```

   (a) What is this function doing?

   (b) What is the asymptotic running time of this function?

   (c) Suggest a simple improvement to this function that guarantees an asymptotic running time of $O(\log n)$.

   (5+5+5)

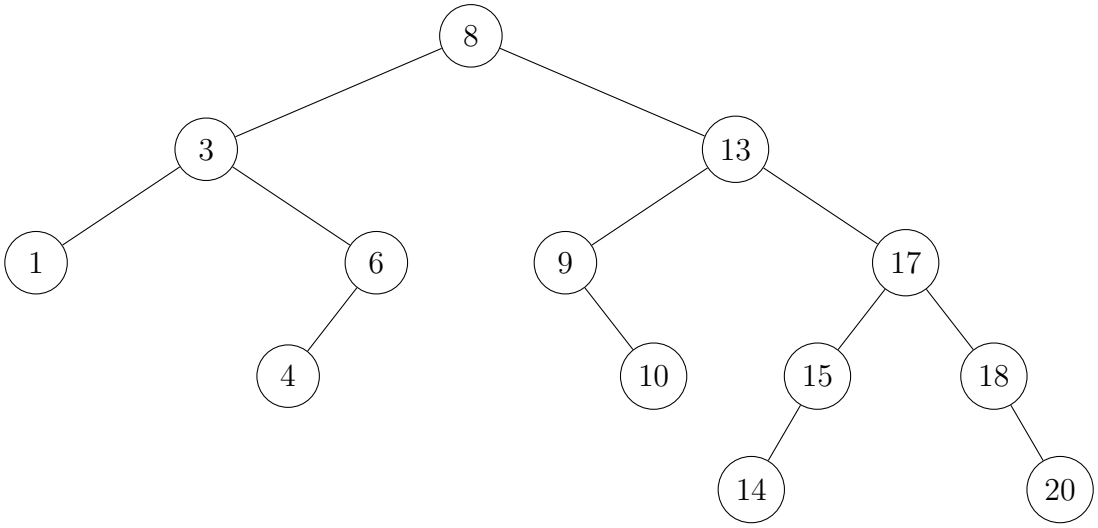
2. Sketch, using Python or Python-like pseudocode, the implementation of a function that computes the height of a tree. (10)

3. (a) Suppose you have array that is sorted in increasing order. Is it a valid array representation of a min-heap?

   (b) You have a valid array representation of a min-heap. Then you reverse the array. Do you get a valid array representation of a max-heap?

   Justify your answer in each case. (5+5)

4. Delete the key 8 from the following AVL tree. Indicate all subtree heights before the deletion, and show the changes introduced by the deletion and subsequent rebalancing step-by-step. (You may consult the summary of tree roations on the front page.)



(10)

5. We noted in class that every implementation of a sorted map can be turned into a sort algorithm as follows: Given a list L to be sorted, take an initially empty sorted map M and move the elements of L one-by-one into the sorted list. Then traverse M in-order, moving the elements back into L.

   For each of the following data structures that can be used to implement a sorted map, state the asymptotic running time in two cases: worst case, and L already sorted. (10)

Implementation	**Worst case**	L **already sorted**
Sorted linked list		
Reverse sorted linked list		
Skip list		
AVL tree		
Splay tree		

6. (a) Show that every rooted tree has at least one leaf.

   (b) For a graph $G = (V, E)$, let $v = |V|$ denote the number of vertices and $e = |E|$ denote the number of edges. Show that G is a tree if and only if

$$v = e + 1\,.$$

   (c) How many connected components[1] does a forest[2] with $v = 20$ vertices and $e = 15$ edges have?

(5+5+5)

[1] A connected component of a graph G is a maximal connected subgraph of G.

[2] A graph G is a forest if it contains no cycles. (So every connected subgraph of a forest is a tree.)