Exercise 6 - Solutions

1. (i) L is smooth, $L(0)=0$, $L(x)>0$ for $x \neq 0$.

(ii) $$\frac{d}{dt} L(x(t)) = \nabla L(x)\cdot \dot{x} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ -\eta x_2 - x_1 \end{pmatrix} = -2\eta x_2^2 \quad , \quad \eta > 0$$

$\Rightarrow$ L is strictly decreasing along orbits ($x_2 = 0$, $x_1 \neq 0$ is not an invariant set!)

So L is a strict Lyapunov function, 0 is asymptotically stable.

Remark: $\dot{x} = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & -\eta \end{pmatrix}$, so $\det(A - \lambda I) = \lambda(\lambda + \eta) + 1$

$$\Rightarrow \lambda_{\pm} = \frac{-\eta \pm \sqrt{\eta^2 - 4}}{2}$$

So linear stability analysis gives the same result here.

2. Convert into first-order system:

$$x_1 = q \quad \Rightarrow \quad \dot{x}_1 = x_2$$
$$x_2 = \dot{q} \qquad \dot{x}_2 = \ddot{q} = -\eta(x_1)x_2 - U'(x_1)$$

(a) Consider the energy $E(x) = \frac{1}{2}x_2^2 + U(x_1)$

$$\Rightarrow \frac{d}{dt} E(x(t)) = \nabla E \cdot \dot{x} = \begin{pmatrix} U'(x_1) \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ -\eta(x_1)x_2 - U'(x_1) \end{pmatrix}$$

$$= -\eta(x_1)\, x_2^2$$

$\Rightarrow$ E is strictly decreasing along orbits unless $x_2 = 0$ on some interval of time, but then $U'(x_1) = 0$ on that interval, so $(0, x_1)$ is an equilibrium point.

$\Rightarrow$ There cannot be a periodic orbit which is not an equilibrium point.

(b) If U has a local minimum at x_i^*, then $U'(x_i^*)=0$ and the differential equation has an equilibrium point at $(x_i^*, 0)$.

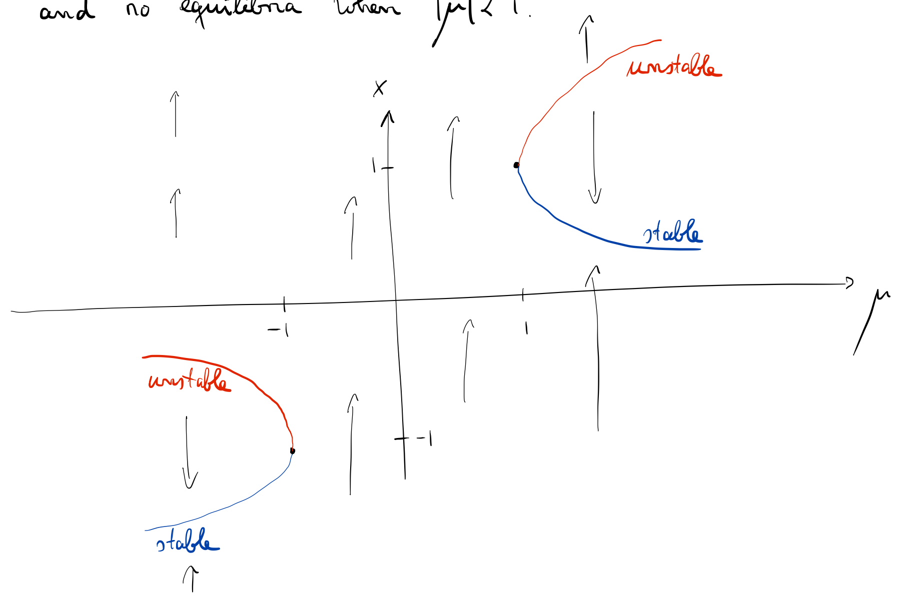
Moreover, $E(x)$ has a local minimum at $(x_i^*, 0)$, so by (a), E is a strict Lyapunov function at $(x_i^*, 0)$.

$\Rightarrow$ $(x_i^*, 0)$ is asymptotically stable.

3. Equilibrium points satisfy $1 - 2\mu x + x^2 = 0$

$$\Rightarrow \quad x_\pm = \frac{2\mu \pm \sqrt{4\mu^2 - 4}}{2} = \mu \pm \sqrt{\mu^2 - 1} \qquad (*)$$

$\Rightarrow$ The equation has pairs of equilibria when $|\mu| > 1$, a single equilibrium when $\mu = \pm 1$ and no equilibria when $|\mu| < 1$.



<u>Note</u>: Candidates for bifurcation points can be found by solving $f'(x) = 0$ where $f(x) = 1 - 2\mu x + x^2$

$\Rightarrow f'(x) = -2\mu + 2x \quad \Rightarrow x = \mu$

with $(*)$, this implies $\mu = \mu \pm \sqrt{\mu^2 - 1} \quad \Rightarrow \mu^2 = 1 \quad \Rightarrow \mu = \pm 1$.

So we have two saddle-node bifurcations.