Exercise 4 - Solutions

1. (a) $x_2 \, (1 + x_1 - x_2^2) = 0 \quad \Rightarrow \quad x_2 = 0 \quad \text{or} \quad 1 + x_1 - x_2^2 = 0$

$x_1 \, (1 + x_2 - x_1^2) = 0 \quad \Rightarrow \quad x_1 = 0 \quad \text{or} \quad 1 + x_2 - x_1^2 = 0$

If $x_2 = 0$: $\quad x_1 = 0 \quad$ or $\quad 1 - x_1^2 = 0 \quad \Rightarrow \quad x_1 = \pm 1$

If $1 + x_1 - x_2^2 = 0$: $\quad x_1 = 0 \quad$ and $\quad x_2 = \pm 1$

or $\quad 1 + x_2 - x_1^2 = 0 \qquad$ with $x_1 = x_2^2 - 1$

$$\Rightarrow 1 + x_2 - x_2^4 + 2x_2^2 - 1 = 0$$

$$\Rightarrow 1 - x_2^3 + 2x_2 = 0 \qquad (\text{since } x_2 \neq 0)$$

By inspection, one root is $x_2 = -1$, so $x_1 = 0$.

Divide out corresponding linear factor: $(x_2^3 - 2x_2 - 1) : (x_2 + 1) = x_2^2 - x_2 - 1$

$$\begin{array}{l}
-)\underline{x_2^3 + x_2^2} \\
\quad -x_2^2 - 2x_2 \\
\underline{-)-x_2^2 - x_2} \\
\qquad -x_2 - 1 \\
\qquad \underline{-x_2 - 1} \\
\qquad\quad 0
\end{array}$$

Thus, the two remaining roots are $x_2 = \dfrac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

Here, $x_1 = x_2^2 - 1 = x_2$!

Thus, we have the following equilibrium points:

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad C' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad D = \frac{1+\sqrt{5}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad E = \frac{1-\sqrt{5}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that the equations are symmetric under exchange $x_1 \Leftrightarrow x_2$, so the points B, B' and C, C' are mirror images of each other about the line $x_1 = x_2$

Now, in general,

$$Df = \begin{pmatrix} x_2 & 1 + x_1 - 3x_2^2 \\ 1 + x_2 - 3x_1^2 & x_1 \end{pmatrix}$$

Let's look at the linear system at each equilibrium point in turn.

At (A) : $Df\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $p(\lambda) = \lambda^2 - 1 \Rightarrow \lambda_{1,2} = \pm 1$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix}1\\1\end{pmatrix} \text{ is unstable direction}$$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix}1\\-1\end{pmatrix} \text{ is stable direction}$$

At (B) : $Df\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & -2\\ 2 & 0\end{pmatrix}$, $p(\lambda) = -\lambda(1-\lambda)+4 = \lambda^2 - \lambda + 4$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-16}}{2}$$

Has positive real part $\Rightarrow$ unstable focus

At (C) : $Df\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}-1 & -2\\ 0 & 0\end{pmatrix}$ $\quad p(\lambda) = (-1-\lambda)(-\lambda) = \lambda(\lambda+1)$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = -1$$

$$\begin{pmatrix}-1 & -2\\ 0 & 0\end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix}2\\-1\end{pmatrix} \quad \text{neutral direction}$$

$$\begin{pmatrix}0 & -2\\ 0 & 1\end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{stable direction}$$

To see what is going on in the neutral direction, write

$$x(t) = \begin{pmatrix}0\\-1\end{pmatrix} + \phi(t)\, v_1 + a\,\phi^2(t)\, v_2 = \begin{pmatrix}2\phi + a\phi^2\\ -1-\phi\end{pmatrix}$$

$$\Rightarrow \dot{x}(t) = \dot{\phi}\, v_1 + 2a\phi\dot{\phi}\, v_2 = \begin{pmatrix}2\dot{\phi} + 2a\phi\dot{\phi}\\ -\dot{\phi}\end{pmatrix}$$

The corresponding RHS reads:

$$\begin{pmatrix}(-1-\phi)(1+2\phi+a\phi^2 - (1+\phi)^2\\ (2\phi+a\phi^2)(1+(-1-\phi)) - (2\phi+a\phi^2)^2\end{pmatrix} = \begin{pmatrix}-(1+\phi)(a-1)\phi^2\\ -2\phi^2 + h.o.t.\end{pmatrix}$$

Thus, consistency requires that $\frac{1-a}{2} = 2 \Rightarrow a = -3$

Then, at leading order the neutral dynamics is given by

$$\dot{\phi} = 2\phi^2 ,$$

i.e., the nonlinear flow is in the direction of v_1

At ⓓ: $$Df\left(\frac{1+\sqrt{5}}{2}\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & -2\left(\frac{1+\sqrt{5}}{2}\right)^2 \\ -2\frac{(1+\sqrt{5})^2}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & -3-\sqrt{5} \\ -3-\sqrt{5} & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

$$p(\lambda) = \left(\frac{1+\sqrt{5}}{2} - \lambda\right)^2 - \left(3+\sqrt{5}\right)^2 = \lambda^2 - \left(1+\sqrt{5}\right)\lambda + \frac{1+2\sqrt{5}+5}{4} - 9 - 6\sqrt{5} - 5$$

$$= \lambda^2 - \left(1+\sqrt{5}\right)\lambda - \frac{25}{2} - \frac{11}{2}\sqrt{5}$$

$$\Rightarrow \lambda_{1,2} = \frac{1+\sqrt{5} \pm \sqrt{1+2\sqrt{5}+5+50+22\sqrt{5}}}{2} = \frac{1+\sqrt{5} \pm \sqrt{56+24\sqrt{5}}}{2}$$

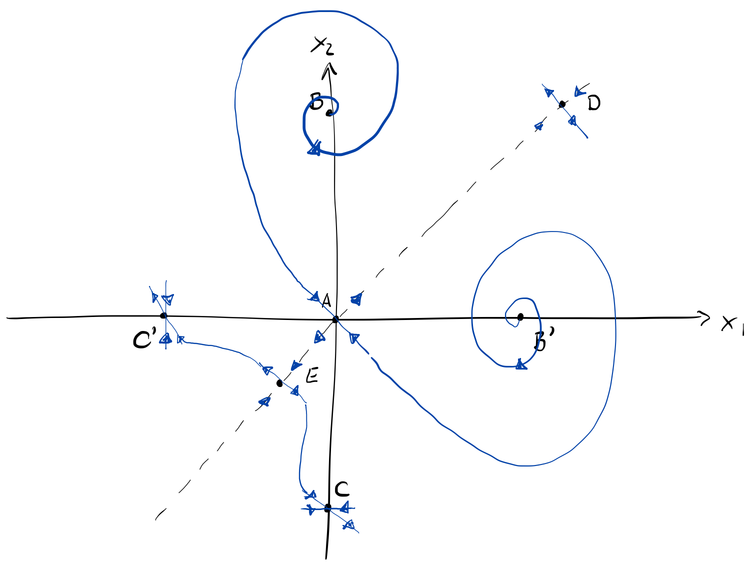
$$\left.\begin{array}{l} \Rightarrow \lambda_1 = \frac{7+3\sqrt{5}}{2}, \quad \lambda_2 = -\frac{5+\sqrt{5}}{2} \\ \text{with corresponding eigenvectors } v_1 = \begin{pmatrix}-1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\1\end{pmatrix} \end{array}\right\} \text{computation not shown}$$

At ⓔ: A similar computation gives

$$\lambda_1 = -\frac{5}{2} + \frac{\sqrt{5}}{2}, \quad v_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\lambda_2 = \frac{7}{2} - \frac{3\sqrt{5}}{2}, \quad v_2 = \begin{pmatrix}-1\\1\end{pmatrix}$$

Phase portrait:



(b) Equilibrium points:

$$2x_1 - x_1^2 - x_1x_2 = 0 \quad \Rightarrow \quad x_1(2 - x_1 - x_2) = 0 \quad \Rightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 2 - x_1$$

$$-x_2 + x_1x_2 = 0 \quad \Rightarrow \quad x_2(x_1 - 1) = 0 \quad \Rightarrow \quad x_2 = 0 \quad \text{or} \quad x_1 = 1$$

Equilibrium points are

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Df = \begin{pmatrix} 2 - 2x_1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{pmatrix}$$

@ A: $Df\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -1$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

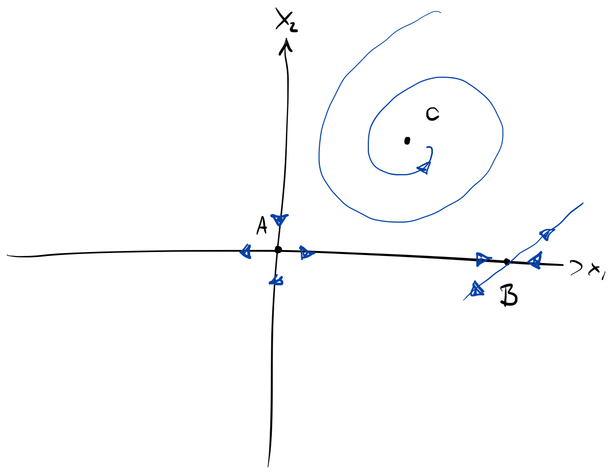
@ B: $Df\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$, $\lambda_1 = -2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = 1$, $v_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$

@ C: $Df\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $p(\lambda) = (1+\lambda)\lambda + 1 = \lambda^2 + \lambda + 1$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\Rightarrow$ stable focus

Phase portrait:



(c) Let $x_1 = y \Rightarrow \dot{x}_1 = x_2$

$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = -\dot{y} - y^3 = -x_2 - x_1^3$ $\qquad \dot{x} = 0 \Rightarrow x_1 = x_2 = 0$

$$\Rightarrow Df = \begin{pmatrix} 0 & 1 \\ -3x_1^2 & -1 \end{pmatrix} \qquad Df\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (neutral direction)

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (stable direction)

To determine dynamics in neutral direction: Let

$$x(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \phi(t)\, v_1 + a\,\phi^3(t)\, v_2 + b\,\phi^5(t)\, v_2$$

$$= \begin{pmatrix} \phi + a\phi^3 + b\phi^5 \\ -a\phi^3 - b\phi^5 \end{pmatrix}$$

$$\Rightarrow \dot{x} = \begin{pmatrix} \dot{\phi}\,(1 + 3a\phi^2 + 5b\phi^4) \\ -\dot{\phi}\,(3a\phi^2 + 5b\phi^4) \end{pmatrix}$$

On the other hand:

$$\dot{x} = \begin{pmatrix} -a\phi^3 - b\phi^5 \\ a\phi^3 + b\phi^5 - (\phi + a\phi^3 + b\phi^5)^3 \end{pmatrix}$$

$$\Rightarrow \dot{\phi}\begin{pmatrix} 1 + 3a\phi^2 + O(\phi^4) \\ -3a - 5b\phi^2 \end{pmatrix} = \begin{pmatrix} -a\phi^3 + O(\phi^5) \\ a\phi + b\phi^3 - \phi - 3a\phi^3 + O(\phi^5) \end{pmatrix}$$

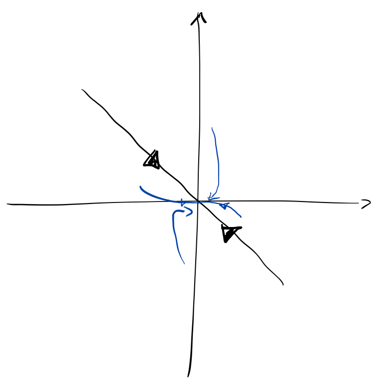
$$\Rightarrow \dot{\phi}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a\phi^3 + O(\phi^5) \\ \left[(a-1)\phi + (b-3a)\phi^3 + O(\phi^5)\right]\left(\frac{1}{-3a} + O(\phi^2)\right) \end{pmatrix}$$

Need $a = 1$, $\frac{b-3a}{-3a} = -1 \Rightarrow \frac{b-3}{3} = 1 \Rightarrow b = 6$

Thus, in the neutral direction the dynamics is given by

$$\dot{\phi} = -\phi^3$$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is (nonlinearly) asymptotically stable.



2. Necessary condition for periodic orbits: Bendixon's criterion.

$$\nabla \cdot f = -3x_1^2 + \mu$$

$\Rightarrow$ periodic orbits can only exist when $\mu > 0$.

The only equilibrium point is $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$Df\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix} \qquad p(\lambda) = (\mu-\lambda)(-\lambda) + 1$$
$$= \lambda^2 - \mu\lambda + 1$$

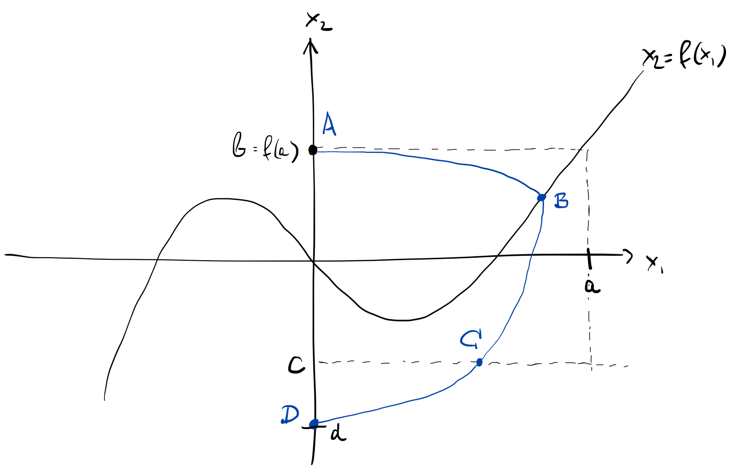
$$\lambda_\pm = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

This equilibrium point is unstable when $\mu > 0$.

It remains to be shown that there is a region around $\binom{0}{0}$ from which orbits cannot leave.

Let's consider the line $\dot{x}_1 = 0 \Rightarrow x_2 = x_1^3 - \mu x_1 \equiv f(x_1)$

- Above this line, $\dot{x}_1 > 0$
- Below this line, $\dot{x}_1 < 0$



Preliminaries:

- fix some point a larger than the right-most root of f
- Set $b = f(a)$
- Fix $c < 0$ s.t. $\dot{x}_1 = x_2 - \frac{1}{3}x_1^3 + \mu x_1 < -1$ for $x_1 \geq 0$, $x_2 \leq c$

(by elementary calculus, for every μ such $c = c(\mu)$ exists.)

Now:

- Consider the orbit starting at $A = (0, b)$
- In this region, $\dot{x}_1 > 0$ and $\dot{x}_2 \leq 0$, so phase point moves South-East until it crosses the graph of $x_2 = f(x_1)$ at some point B.
- On the other side, $\dot{x}_1 < 0$ and $\dot{x}_2 < 0$, so phase point moves South-West and either hits the x_2-axis without crossing the line $x_2 = c$ or crosses that line at some point C.

In the first case, the phase point hits the y-axis at a point $D = (0, d)$ with $d \geq c$.

In the second case, note that $C = (\gamma, c)$ with $\gamma \leq a$.

For the orbit from C to D, we have $x_2 \leq c$, so

$$\left.\begin{matrix} \dot{x}_1 \leq -1 \\ \dot{x}_2 = -x_1 \end{matrix}\right\} \quad \frac{dx_2}{dx_1} \geq x_1 \Rightarrow \int_c^d dx_2 \geq \int_\gamma^0 x_1\, dx_1$$

$$\Rightarrow d - c \geq \tfrac{1}{2} x_1^2 \Big|_\gamma^0 = -\tfrac{1}{2}\gamma^2 \geq -\tfrac{1}{2}a^2$$

$$\Rightarrow d \geq c - \tfrac{1}{2}a^2$$

On the other hand, $b = a^3 - \mu a$

$$\Rightarrow -d \leq \tfrac{1}{2}a^2 - c \leq b$$

for a sufficiently large

Conclusion: If a is large enough, the orbit coming out of A hits the x_2-axis at a point D closer to the origin. Since the phase portrait is point symmetric about the origin, the same can be said for the orbit coming out of $A' = (0, -b)$. Thus, orbits cannot leave the box $[-a, a] \times [-b, b]$.

By the Poincaré-Bendixon theorem, there must be a periodic orbit.