Exercise 3 - Solutions

1. Let $J_i = \lambda_i I + N_i$, $N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \Big\}\, k_i$ rows

denote the i-th Jordan block. We know that $\mathrm{Re}\,\lambda_i \leq \alpha$. Then, from class:

$$e^{J_i t} = e^{\lambda_i t} \sum_{j=0}^{k_i-1} \frac{N_i^j t^j}{j!}$$

$$\Rightarrow \| e^{J_i t} \| \leq \underbrace{|e^{\lambda_i t}|}_{= e^{\mathrm{Re}\lambda_i t} \leq e^{\alpha t}} \sum_{j=0}^{k_i-1} \frac{t^j}{j!} \|N_i\|^j$$

where $\|N_i\| = \sup_{\|x\|=1} \|N_i x\| \leq 1$

(for any isotropic vector norm, otherwise bound by a constant ...)

$$\leq e^{\alpha t} \sum_{j=0}^{k_i-1} \frac{t^j}{j!} \qquad (*)$$

In general, if $A = SJS^{-1}$ with $J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\ell \end{pmatrix}$,

$$e^{At} = S e^{Jt} S^{-1}$$

$$\Rightarrow \| e^{At} \| \leq \|S\| \, \|e^{Jt}\| \, \|S^{-1}\| \qquad (**)$$

where
$$\|e^{Jt}\| = \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \|e^{Jt} x\|$$

$$= \max_{i \in 1,\dots,\ell} \max_{\substack{x \in \mathbb{R}^{k_i} \\ \|x\|=1}} \|e^{J_i t} x\|$$ since Jordan subspaces are orthogonal

$$= \max_{i \in 1,\dots,\ell} \|e^{J_i t}\|$$

$$\overset{(*)}{\leq} \max_{i \in 1,\dots,\ell} e^{\alpha t} \sum_{j=0}^{k_i-1} \frac{t^j}{j!}$$

$$= e^{\alpha t} \sum_{j=0}^{k-1} \frac{t^j}{j!}$$ with $k = \max_{i \in 1,\dots,\ell} k_i$

Plugging into (**) proves the claim.

<u>Remark</u>: since $t \leq \frac{1}{2}(1+t^2)$

$$t^2 \leq t\,\tfrac{1}{2}(1+t^2) \leq \tfrac{1}{4}(1+t^2) + \tfrac{1}{2}t^3 \Rightarrow \tfrac{3}{4}t^2 \leq \tfrac{1}{4} + \tfrac{1}{2}t^3 \Rightarrow t^2 \leq \tfrac{1}{3} + \tfrac{2}{3}t^3$$

etc.

we can estimate all intermediate powers of t in terms of 1 and t^{k-1}.

Remark 2: The estimate reads

$$\|e^{At}\| \leq e^{\alpha t} p(t) \qquad \text{for some polynomial } p \text{ with non-negative coefficients.}$$

$$\Rightarrow \quad \|e^{At}\| \leq e^{(\alpha+\varepsilon)t} \, e^{-\varepsilon t} \, p(t) \qquad \text{for any } \varepsilon > 0$$

$$\leq e^{(\alpha+\varepsilon)t} \underbrace{\max_{t \in [0,\infty)} e^{-\varepsilon t} p(t)}$$

this expression is finite, but will depend on ε.

$$= C(\varepsilon) \, e^{(\alpha+\varepsilon)t}$$

This is the bound we used to prove the asymptotic stability theorem in class.

2. (a) Let $x_1 = q$, $x_2 = \dot{q}$

$$\left.\begin{aligned} \Rightarrow \quad \dot{x}_1 &= \dot{q} = x_2 \\ \dot{x}_2 &= \ddot{q} = -\dot{q} - q = -x_1 - x_2 \end{aligned}\right\} \Rightarrow \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}_{=:A} x$$

(b) Characteristic polynomial $p(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda(1+\lambda) + 1 = \lambda^2 + \lambda + 1$

Roots: $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$

$\Rightarrow$ A is diagonalizable, all Jordan blocks have size $k=1$.

$\Rightarrow$ The estimate from Problem 1 applies with $\alpha = -\frac{1}{2}$

(c) Let's parametrize unit vectors $x = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Then

$$x^T A x = (\cos\theta \;\; \sin\theta) \begin{pmatrix} \sin\theta \\ -\cos\theta - \sin\theta \end{pmatrix} = \sin\theta\cos\theta - \cos\theta\sin\theta - \sin^2\theta$$

$$= -\sin^2\theta \leq 0 \qquad \text{with equality for } \theta = n\pi, \; n \in \mathbb{Z}$$

(Physically, this means the displacement is maximal, the velocity is 0.)

(d) $\underbrace{x^T \dot{x}}_{= \frac{1}{2}\dot{E}} = x^T A x \leq 0 \quad \Rightarrow$ The energy is decreasing, but not strictly decreasing

(e) Damping is proportional to velocity, so at times when the velocity is 0, the rate of energy "loss" is also zero.

3. Write $A = SJS^{-1}$

$$\Rightarrow \quad S^{-1}\dot{x} = JS^{-1}x \quad \text{or} \quad \dot{y} = Jy \quad \text{with} \quad x = Sy$$

Let's focus on a single Jordan block J_i of size $k_i > 1$.

$$e^{J_i t} = e^{\lambda_i t}\left(I + \dots + \frac{1}{(k_i-1)!} N_i^{k_i-1}\right)$$

Take $y_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow e^{J_i t} y_0 = e^{\lambda_i t}\left(y_0 + t\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = e^{\lambda_i t}\begin{pmatrix} t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\Rightarrow \|x(t)\|^2 = \left\| e^{\lambda_i t} S_i \begin{pmatrix} t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|^2 = e^{2\mathrm{Re}\,\lambda_i t} \| t v_1 + v_2 \|^2 = e^{2\mathrm{Re}\,\lambda_i t}\left(t^2 + \|v_2\|^2\right)$$

Thus, if $\mathrm{Re}\,\lambda_i < 0$ and $|\mathrm{Re}\,\lambda_i|$ is small enough in relation to $\|v_2\|^2$, this expression will have an interval of growth for positive times.