Exercise 2 - Solutions

1. Solution A (by direct computation):

$$E = \frac{1}{2}\dot{x}^2 + F(x) \quad \Rightarrow \quad \dot{E} = \dot{x}\ddot{x} + F'(x)\dot{x}$$
$$= -\dot{x}f(x) + f(x)\dot{x} = 0$$

Solution B (via first-order system):

write $q = x$, $p = \dot{x}$, $y = \begin{pmatrix} q \\ p \end{pmatrix} \quad \Rightarrow \quad \dot{y} = \begin{pmatrix} p \\ -f(q) \end{pmatrix}$

$$\Rightarrow \dot{y} \cdot \begin{pmatrix} f(q) \\ p \end{pmatrix} = 0 \quad \Rightarrow \quad \dot{q}f(q) + \dot{p}p = 0 \quad \Rightarrow \quad \frac{d}{dt}\underbrace{\left(F(q) + \frac{1}{2}p^2\right)}_{=:E} = 0$$

2. (a) Equilibrium points are solutions to

$$\begin{cases} x - xy = 0 & \Rightarrow x(1-y) = 0 \quad \Rightarrow x = 0 \text{ or } y = 1 \\ xy - y = 0 & \Rightarrow y(x-1) = 0 \quad \Rightarrow y = 0 \text{ or } x = 1 \end{cases}$$

Only $\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ lies in the interior of the first quadrant.

(b) Write $x = 1 + \xi$ and $y = 1 + \eta$, then insert into equations:

$$\dot{\xi} = \dot{x} = x - xy = x(1-y) = (1+\xi)(1-1-\eta) = -(1+\xi)\eta = -\eta - \xi\eta$$
$$\dot{\eta} = \dot{y} = y(x-1) = (1+\eta)\xi = \xi + \xi\eta$$
$$\Rightarrow \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \text{h.o.t.}$$

(c) Solution A (via Problem 1): $\ddot{\xi} = -\dot{\eta} = -\xi \Rightarrow \ddot{\xi} + \xi = 0$

so $f(\xi) = \xi$ and $F(\xi) = \frac{1}{2}\xi^2$

Then, by Problem 1, $E = \frac{1}{2}\dot{\xi}^2 + F(\xi) = \frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\xi^2 = \frac{1}{2}\eta^2 + \frac{1}{2}\xi^2$ is a constant of the motion

$\Rightarrow$ The trajectories lie on circles, i.e. remain bounded.

Solution B (direct "energy estimate"):

Let $z = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Linear equation: $\dot{z} = Az$ where A is skew, i.e. $A = -A^T$.

$$\Rightarrow \underbrace{z^T\dot{z}}_{= \frac{1}{2}\frac{d}{dt}\|z\|^2} = z^T A z = 0 \quad \Rightarrow \quad \|z(t)\| = \text{const}, \text{ hence does not grow as } t \to \infty.$$