Exercise 1 – Solutions

1. Let $x_1 = \theta$, $x_2 = \dot{\theta}$

$\Rightarrow \dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{\theta} = -\sin\theta = -\sin x_1$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $f(x) = \begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix}$ $\Rightarrow$ $\dot{x} = f(x)$

2. (a) Integrating factor $M = e^{\int_0^t (-1)\,ds} = e^{-t}$

$$\Rightarrow \frac{d}{dt}\left(e^{-t}x\right) = 2t\, e^{2t} e^{-t} = 2t\, e^{t}$$

$$\Rightarrow e^{-t}x(t)\Big|_0^t = \int_0^t 2s\, e^{s}\, ds$$

$$\Rightarrow e^{-t}x(t) - \underbrace{e^{-0}x(0)}_{=1} = 2s\, e^{s}\Big|_0^t - \int_0^t 2e^{s}\, ds$$

$$= 2t e^{t} - 2\cdot 0\, e^{0} - 2e^{s}\Big|_0^t$$

$$= 2t e^{t} - 2e^{t} + 2$$

$$\Rightarrow e^{-t}x(t) = 2t\, e^{t} - 2e^{t} + 3$$

$$\Rightarrow x(t) = 2t\, e^{2t} - 2e^{2t} + 3e^{t}$$

(b) $t\dot{x} + 2x = t^2 - t + 1$

$$\Rightarrow \dot{x} + \frac{2}{t}x = t - 1 + t^{-1}$$

Integrating factor: $M = e^{\int_1^t \frac{2}{s} ds} = e^{2(\ln t - \ln 1)} = t^2$

$$\Rightarrow \frac{d}{dt}\left(t^2 x\right) = t^3 - t^2 + t$$

$$\Rightarrow t^2 x(t) - 1^2 x(1) = \int_1^t (s^3 - s^2 + s)\, ds = \frac{1}{4}s^4 - \frac{1}{3}s^3 + \frac{1}{2}s^2\Big|_1^t$$

$$\Rightarrow t^2 x(t) - \frac{1}{2} = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$$

$$\Rightarrow t^2 x(t) = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{1}{12}$$

$$\Rightarrow x(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12}t^{-2}$$

(c) $(3+4x)\dot{x} = e^{-t} - e^{t}$

$$\Rightarrow \int_{x(0)}^{x(t)} (3+4x)\,dx = \int_0^t (e^{-s} - e^{s})\,ds$$

$$\Rightarrow 3x + 2x^2 \Big|_{x(0)}^{x(t)} = -e^{-s} - e^{s} \Big|_0^t$$

$$\Rightarrow 3x(t) + 2x^2(t) - 3 - 2 = -e^{-t} - e^{t} + 2$$

$$\Rightarrow 2x^2(t) + 3x(t) + e^{t} + e^{-t} - 7 = 0$$

$$\Rightarrow x(t) = \frac{-3 \pm \sqrt{9 - 8(e^{t} + e^{-t} - 7)}}{4}$$

Note: $x(0) = \dfrac{-3 \pm \sqrt{9 + 8\cdot 5}}{4} = \dfrac{-3 \pm 7}{4}$

To get $x(0) = 1$, need to choose + - sign!

3. (a) $U = xy$

$$\Rightarrow \dot{U} = \dot{x}y + x\dot{y}$$

$$= (x - xy)y + x(xy - y)$$

$$= xy(x-y) = U(x-y)$$

$$\Rightarrow \underbrace{\int_{U(0)}^{U(t)} \frac{dU}{U}}_{= \ln \frac{U(t)}{U(0)}} = \int_0^t (x-y)\,ds$$

$$\Rightarrow u(t) = u(0)\, e^{\overbrace{\int_0^t (x-y)\,ds}}$$
$$\underbrace{}_{>0}$$

$\Rightarrow$ If $u(0) \neq 0$, $\quad u(t) > 0 \quad \forall\, t \geq 0$

So neither $x(t)$ nor $y(t)$ can change sign.

(b) $\dot{v} = \dot{x} + \dot{y} = x - xy + xy - y = x - y$

$$\Rightarrow \dot{v} \leq v \quad \text{if } xy > 0$$

$$\Rightarrow \int_{v(0)}^{v(t)} \frac{dv}{v} \leq \int_0^t 1\, ds$$

$$\Rightarrow \ln \frac{v(t)}{v(0)} \leq t$$

$$\Rightarrow v(t) \leq v(0)\, e^t$$

So $v(t)$ cannot blow up in finite time.

(c) $0 \leq (a-b)^2 = a^2 - 2ab + b^2$

$$\Rightarrow 2ab \leq a^2 + b^2$$

Set $a = \sqrt{x}$, $\; b = \sqrt{-y}$, $\quad w = -xy \qquad x > 0,\; y < 0$

$$\Rightarrow 2\sqrt{w} = 2ab \leq a^2 + b^2 = x - y$$

Now $\dot{w} = -\dot{x}y - y\dot{x} = -xy\,(x-y) = w\,(x-y) \geq 2w^{\frac{3}{2}}$

$$\Rightarrow \int_{w(0)}^{w(t)} w^{-\frac{3}{2}}\, dw \geq 2t \Rightarrow -2w^{-\frac{1}{2}}\Big|_{w(0)}^{w(t)} \geq 2t \Rightarrow \frac{2}{\sqrt{w(0)}} - \frac{2}{w(t)} \geq 2t$$

$$\Rightarrow \frac{1}{w(t)} \leq \frac{1}{w(0)} - t \Rightarrow w(t) \geq \frac{1}{\frac{1}{w(0)} - t}$$

which blows up latest when $t = \frac{1}{w(0)}$!