Differential Equations

Summer Semester 2024, Exercise 6

Due Wednesday, June 19, 2024

1. (From Teschl, Problem 6.18.) Show that $L(x) = x_1^2 + x_2^2$ is a Lyapunov function for the system

$$\dot{x}_1 = x_2\,,$$
$$\dot{x}_2 = -\eta\, x_2 - x_1$$

where $\eta \geq 0$ in a neighborhood of the equilibrium point $x^* = (0,0)$. What can you say about the stability of the equilibrium point?

2. (From Teschl, Problem 6.27.) Consider a general system with friction of the form

$$\ddot{q} = -\eta(q)\,\dot{q} - U'(q)$$

with $\eta(q) > 0$ and $U(q)$ some differentiable function with Lipschitz continuous derivative $U'(q)$.

(a) Show that there are no non-trivial periodic solutions.
Hint: Use the energy.

(b) Show that minima of U correspond to asymptotically stable equilibria in phase space.

3. Study the stability of the equilibria of

$$\dot{x} = 1 - 2\mu x + x^2$$

as a function of μ. Which bifurcations occur?

4. A very simple global energy balance model for planet Earth reads

$$C\,\dot{T} = (1 - \alpha(T))\,Q - \varepsilon\,\sigma\,T^4$$

where C is the heat capacity of the planet, the precise value does not matter here, $\sigma = 5.67 \cdot 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is the so-called Stefan constant (the corresponding term in the equation describes the energy emitted from Earth via black-body radiation per area), $\varepsilon = 0.6$ a "Greenhouse effect" constant that models that parts of the black-body spectrum are reflected back to the surface, Q is the solar energy reaching Earth per area with a current value of $Q_0 = 342\,\mathrm{W\,m^{-2}}$, and α is the albedo, i.e., the fraction of solar radiation reflected back into space. It is temperature dependent and assumed to follow the profile

$$\alpha(T) = 0.5 - 0.2 \tanh \frac{T - 265\,\mathrm{K}}{10\,\mathrm{K}}$$

which models a smooth transition from full ice cover when $T \ll 273\,\mathrm{K}$ so that the albedo is high to a surface without any ice when $T \gg 273\,\mathrm{K}$ so that albedo is low.[1] In the following, we shall use Q/Q_0 as the bifurcation parameter.

(a) Use Matplotlib to plot the location of equilibria in a diagram with Q/Q_0 on the horizontal and T on the vertical axis.

(b) Describe the approximate location and type of bifurcations.

(c) Which branches of equilibria are stable, which are unstable?

[1] Details about this and other models can, for example, be found in H. Kaper, H. Engler, *Mathematics and Climate*, SIAM, 2013.