Differential Equations

Summer Semester 2024, Exercise 5

Due Wednesday, May 29, 2024

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= a\,x_1 + b\,x_2 - x_1^2\,x_2 - x_1^3 \,. \end{aligned}$$

   Show that there cannot be a periodic orbit unless $b > 0$.

   Hint: Use Bendixon's criterion.

2. Suppose γ_1 and γ_2 are two periodic orbits with γ_2 in the interior of γ_1. Suppose further that there are no critical points or periodic orbits in the annular region A between γ_1 and γ_2. Show that, for $x \in A$, $\omega(\gamma(x))$ is either γ_1 or γ_2. Show further that the ω-limitset is the same for all orbits in A.

   Hint: Poincaré–Bendixon theorem.

3. (From Verhulst, Exercise 4.8.) Consider the system

$$\begin{aligned} \dot{x} &= \frac{\partial E}{\partial y} + \lambda\, E\, \frac{\partial E}{\partial x} \,, \\ \dot{y} &= -\frac{\partial E}{\partial x} + \lambda\, E\, \frac{\partial E}{\partial y} \end{aligned}$$

   with $\lambda \in \mathbb{R}$ and

$$E(x, y) = y^2 - 2\,x^2 + x^4 \,.$$

   (a) Put $\lambda = 0$. Determine the critical points and their character by linear analysis. What happens in the nonlinear system? Sketch the phase plane.

   (b) What happens to the critical points when $\lambda \neq 0$?

   (c) Choose $\lambda < 0$. Determine the ω-limit sets of the orbits starting at $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$, and $(1, 2)$.