# Differential Equations

Summer Semester 2024, Exercise 3

Due Monday, May 6, 2024

1. Let $A \in \mathbb{R}^{n\times n}$ such that $\operatorname{Re}\lambda \le \alpha$ for every eigenvalue $\lambda$ of $A$. Let $k$ be the dimension of the biggest Jordan block in the Jordan canonical form of $A$. Show that there exists a constant $C$ such that

$$\|\mathrm{e}^{At}\| \le C\,\mathrm{e}^{\alpha t}\left(1 + t + \cdots + \frac{t^{k-1}}{(k-1)!}\right).$$

   *Hint:* First prove the result for a single Jordan block (using the solution formula from class; also see Teschl, equation 3.18). Then argue for general $A$.

   *Note:* This bound immediately implies the upper bound

$$\|\mathrm{e}^{At}\| \le \tilde{C}\,\mathrm{e}^{\alpha t}\,(1 + t^{k-1})$$

   for some other constant $\tilde{C}$. (Why? – For discussion only.)

2. Consider the damped harmonic oscillator in the form

$$\ddot{q} + \dot{q} + q = 0\,.$$

   (a) Show that this equation can be written in the form

$$\dot{x} = Ax$$

   with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

   (b) Show that $A$ satisfies the assumption of Problem 1 above. What is $\alpha$, what is $k$?

   (c) Show that $A$ is *not* strictly negative definite. (A matrix $A \in \mathbb{R}^{n\times n}$ is strictly negative definite if $x^T A x < 0$ for all $x \in \mathbb{R}^n$.)

   (d) Show that rate of change of the energy $E = x^T x$ is not strictly negative at all times.

   (e) Give a physical interpretation of this fact. Moreover, argue that it is nonetheless consistent with the conclusion of Problem 2.

3. Show that if all eigenvalues of a matrix $A$ have negative real part, but $A$ is not diagonalizable (i.e., its Jordan canonical form has at least one block of dimension $k > 1$), then the corresponding differential equation

$$\dot{x} = Ax$$

has "transient growth", i.e., there are initial conditions such that $E = x^T x$ is growing on some interval of time, even through (by the result from Problem 1 above) $E(t) \to 0$ as $t \to \infty$.