Differential Equations

Summer Semester 2024, Exercise 2

Due Monday, April 29, 2024

1. Consider a scalar second-order differential equation of the form

$$\ddot{x} + f(x) = 0$$

and suppose that F is an anti-derivative of f, i.e., that $F'(x) = f(x)$. Show that the "energy"

$$E = \tfrac{1}{2}\,\dot{x}^2 + F(x)$$

is a constant of the motion, i.e., that $\dot{E} = 0$.

2. Recall the Volterra–Lotka system from Exercise 1, again with all coefficients set to one,

$$\dot{x} = x - x\,y\,,$$
$$\dot{y} = x\,y - y\,.$$

(a) Show that $(x^*, y^*) = (1, 1)$ is the only equilibrium point in the open first quadrant.

(b) Letting $x(t) = x^* + \xi(t)$ and $y(t) = y^* + \eta(t)$, show that for small perturbations ξ and η, the equations take the form

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \text{higher order terms}$$

where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

(c) Show that for the linearized system, i.e., ignoring the higher-order terms in the expression from (b), the amplitude of the perturbation ξ and η does not grow as $t \to \infty$.

Hint: There are different ways to do this. One option is to observe that the linearized system is in fact a particular case of Problem 1 by an appropriate change of notation (cf. Problem 1 of Exercise 1). What is f, what is F in this case? What can you conclude from the the statement that E is a constant of the motion?