Differential Equations

Summer Semester 2024, Exercise 1

Due Wednesday, April 24, 2024

1. Recall the equation for the mathematical pendulum,

$$\ddot{\theta} = -\sin\theta\,.$$

   Convert this equation into a first-order system of the form

$$\dot{x} = f(x)\,.$$

2. Find the solution of the given initial value problem.

   (a) $x' - x = 2t\mathrm{e}^{2t}, \quad x(0) = 1$

   (b) $tx' + 2x = t^2 - t + 1, \quad x(1) = \frac{1}{2}, \quad t > 0$

   (c) $x' = (\mathrm{e}^{-t} - \mathrm{e}^{t})/(3 + 4x), \quad x(0) = 1$

3. Consider the Volterra–Lotka system, here with all coefficients set to one,

$$\dot{x} = x - x\,y\,,$$
$$\dot{y} = x\,y - y\,.$$

   (a) Show that when $x > 0$ and $y > 0$ at time $t = 0$, the solutions remain strictly positive for as long as they exist.
   Hint: What equation is satisfied by $u = xy$?

   (b) Show that positive solutions exist for all times.
   Hint: What equation is satisfied by $v = x + y$?

   (c) What can you say when you drop the condition of positivity of the initial values?
   Hint: Suppose $x > 0$ and $y < 0$. First, set $w = -xy$. Show that

$$2\sqrt{w} \le x - y\,.$$

   What equation is satisfied by w?