Differential Equations

Mock Exam

July 8, 2024

1. Solve the scalar differential equation

$$\dot{x} - \frac{2}{t}\, x = t^2 \cos t\,,$$
$$x(\pi/2) = 0\,.$$

(5)

2. Find the general solution to the two-dimensional system

$$\dot{x} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} x\,.$$

(10)

3. Consider the system

$$\dot{x}_1 = (2 - x_1^2 - x_2^2)\, x_1\,,$$
$$\dot{x}_2 = x_1 - x_2\,.$$

   (a) Find and classify all equilibrium points.

   (b) Sketch the phase portrait. If there are centers or foci, make sure that you the the orientation right.

(5+5)

4. Consider the system

$$\dot{x}_1 = -x_2 - x_1^3\,,$$
$$\dot{x}_2 = x_1^5\,.$$

   (a) Show that $(0,0)$ is the only equilibrium point. What can you say by linear stability analysis?

   (b) Find a strict Lyapunov function and conclude that $(0,0)$ is asymptotically stable.

   Hint: Multiply the first equation with some power of x_1 and the second equation with some power of x_2. Adjust powers and pre-factors such that indefinite terms cancel when you add both. (5+5)

5. Consider the scalar equation with parameter r,

$$\dot{x} = r\,x + x^2 - x^3 \,.$$

Find and classify all bifurcations. (10)

6. Consider a scalar differential equation of the form

$$\dot{x} = f(x)\,,$$
$$x(0) = x_0$$

and assume that $x(t) \leq c$ on some time interval $t \in [0, T]$. Suppose that $y(t)$ solves a modified equation with same initial condition,

$$\dot{y} = f(y) + \cos(\omega t)\,,$$
$$y(0) = x_0\,.$$

Show that $x(t)$ and $y(t)$ are close as $\omega \to \infty$, more precisely, that

$$z(t) = y(t) - x(t) = O(\omega^{-1})$$

for $t \in [0, T]$.

Hints: Derive an equation for $z^2(t)$ and integrate by parts. Further, recall the Gronwall inequality: If $C > 0$ and ϕ and ψ are continuous, non-negative, and satisfy

$$\phi(t) \leq \int_0^t \psi(s)\,\phi(s)\,\mathrm{d}s + C\,,$$

on $[0, T]$, then

$$\phi(t) \leq C\,\exp\biggl(\int_0^t \psi(s)\,\mathrm{d}s\biggr)\,.$$

(5)