1. Solve the scalar differential equation

$$\dot{x} - \frac{2}{t}x = t^2 \cos t \,,$$
$$x(\pi/2) = 0 \,.$$

(5)

Integrating factor:

$$M = \exp\Big(-2 \underbrace{\int_1^t \frac{1}{s}\,ds}_{= \ln t}\Big)$$

(Bottom limit is arbitrary, choose for convenience.)

$$= t^{-2}$$

$$\Rightarrow \frac{d}{dt}\left(t^{-2} x(t)\right) = t^{-2} t^2 \cos t = \cos t$$

$$\Rightarrow t^{-2} x(t) \Big|_{\frac{\pi}{2}}^{t} = \underbrace{\int_{\frac{\pi}{2}}^{t} \cos s \,ds}_{= \sin t - \sin\frac{\pi}{2} = \sin t - 1}$$

$$\Rightarrow t^{-2} x(t) - \left(\frac{2}{\pi}\right)^2 \underbrace{x\left(\frac{\pi}{2}\right)}_{=0} = \sin t - 1$$

$$\Rightarrow x(t) = t^2 (\sin t - 1)$$

Check (not required, but useful): $x\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \left(\sin\frac{\pi}{2} - 1\right) = 0$ ✓

$$\dot{x}(t) = 2t(\sin t - 1) + t^2 \cos t$$

$$\Rightarrow \dot{x} - \frac{2}{t}x = 2t(\sin t - 1) + t^2\cos t - \frac{2}{t}t^2(\sin t - 1) = t^2 \cos t$$ ✓

2. Find the general solution to the two-dimensional system

$$\dot{x} = \underbrace{\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}}_{=:A} x\,. \tag{10}$$

Let's put A into normal form:

$$p_A(\lambda) = \det(A - \lambda I) = (-3-\lambda)(-1-\lambda) + 1 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$\Rightarrow$ There is an eigenvalue $\lambda = -2$ with algebraic multiplicity 2.

For eigenvectors, solve $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (up to const...)

gen. eigenvector: $(A - \lambda I)w = v \Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (other choices possible)

$\Rightarrow$ change of coordinates is $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ with $S^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$$J = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

(note: this you know already, so this computation is just a check!)

General solution:

$$x(t) = e^{At} x_0 = S e^{Jt} S^{-1} x_0 \quad \text{with} \quad e^{Jt} = e^{-2t}\left(I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right)$$

$$= \underbrace{e^{-2t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{=\begin{pmatrix} 1-t & t \\ -1 & 1 \end{pmatrix}}}_{=\begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix}} x_0 = e^{-2t}\begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} x_0$$

check (not required, but useful): $\dot{x}(t) = -2e^{-2t}\begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} x_0 + e^{-2t}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} x_0$

while $Ax = e^{-2t}\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} x_0 = e^{-2t}\begin{pmatrix} -3+2t & -2t+1 \\ -1+2t & -2t-1 \end{pmatrix} x_0$

coincide!

3. Consider the system

$$\dot{x}_1 = (2 - x_1^2 - x_2^2)\,x_1\,,$$
$$\dot{x}_2 = x_1 - x_2\,.$$

(a) Find and classify all equilibrium points.

(b) Sketch the phase portrait. If there are centers or foci, make sure that you the the orientation right.

(5+5)

(a) $\left.\begin{array}{l}(2-x_1^2-x_2^2)\,x_1 = 0 \\ x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = x_2\end{array}\right\} \Rightarrow 2(1-x_1^2)\,x_1 = 0 \Rightarrow x_1 = 0, \quad x_1 = \pm 1$

$\Rightarrow$ equilibrium points are $(0,0)$, $(1,1)$, $(-1,-1)$

$$Df = \begin{pmatrix} 2-x_1^2-x_2^2-2x_1x_1 & -2x_2x_1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2-3x_1^2-x_2^2 & -2x_1x_2 \\ 1 & -1 \end{pmatrix}$$

@ $(0,0)$: $Df = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \Rightarrow$ eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -1$

eigenvectors: $(Df - \lambda_1 I)v_1 = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$(Df - \lambda_2 I)v_2 = 0 \Rightarrow \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

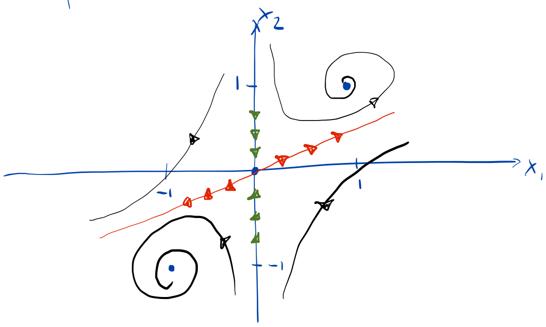
This is a saddle, v_1 is the unstable direction, v_2 the stable direction

@ $(1,1)$: $Df = \begin{pmatrix} -2 & -2 \\ 1 & -1 \end{pmatrix}$, $\quad p(\lambda) = (-2-\lambda)(-1-\lambda) + 2 = \lambda^2 + 3\lambda + 4$

$$\Rightarrow \lambda_\pm = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm \frac{i}{2}\sqrt{7}$$

this is a stable focus, calculation at $(-1,-1)$ is identical.

To determine the direction of the spiral, look at line $x_2 = 1$. Then $\dot{x}_2 = x_1 - 1 > 0$ for $x_1 > 1$
$\Rightarrow$ spiral is anticlockwise. At $x_2 = -1$, $\dot{x}_2 = x_1 + 1 > 0$ for $x_1 > -1$, also anticlockwise.



4. Consider the system

$$\dot{x}_1 = -x_2 - x_1^3\,,$$
$$\dot{x}_2 = x_1^5\,.$$

(a) Show that $(0,0)$ is the only equilibrium point. What can you say by linear stability analysis?

(b) Find a strict Lyapunov function and conclude that $(0,0)$ is asymptotically stable.

Hint: Multiply the first equation with some power of x_1 and the second equation with some power of x_2. Adjust powers and pre-factors such that indefinite terms cancel when you add both. (5+5)

(a)
$$\left.\begin{array}{l} -x_2 - x_1^3 = 0 \\ x_1^5 = 0 \quad \Rightarrow x_1 = 0 \end{array}\right\} \Rightarrow x_2 = 0$$

Linear system about the origin has matrix $Df\big|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

System has eigenvalue $\lambda = 0$ with algebraic multiplicity 2, so is linearly neutral.

(b)
$$x_1^p \dot{x}_1 = -x_1^p x_2 - x_1^{3+p}$$
$$x_2^q \dot{x}_2 = x_1^5 x_2^q$$

terms coincide for $p=5$, $q=1$

Then
$$x_1^5 \dot{x}_1 = -x_1^5 x_2 - x_1^8$$
$$+)\quad x_2 \dot{x}_2 = x_1^5 x_2$$

$$\underbrace{x_1^5 \dot{x}_1 + x_2 \dot{x}_2} = -x_1^8$$
$$= \frac{d}{dt}\left(\frac{1}{6}x_1^6 + \frac{1}{2}x_2^2\right)$$

So $V = \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2$ is a candidate Lyapunov function.

Check: V is cont., $V \geq 0$, $V(0,0) = 0$, V can only be zero if $x_1 = x_2 = 0$

$$\dot{V} = -8x_1^8 \leq 0$$

moreover, V is strictly decreasing along orbits: The flow is transverse to the line $x_1 = 0$ where $\dot{V} = 0$, as for $x_1 = 0$, $\dot{x}_1 = -x_2$, so it's $\neq 0$ if $x_2 \neq 0$.

Thus, V is a strict Lyapunov function $\Rightarrow$ $(0,0)$ is asymptotically stable.

5. Consider the scalar equation with parameter r,

$$\dot{x} = r\,x + x^2 - x^3\,.$$

Find and classify all bifurcations. (10)

Equilibrium points: $rx + x^2 - x^3 = 0 \Rightarrow x(r + x - x^2) = 0$

$\Rightarrow x = 0$ or $x = \frac{1}{2} \pm \sqrt{\frac{1}{4} + r}$ $(*)$

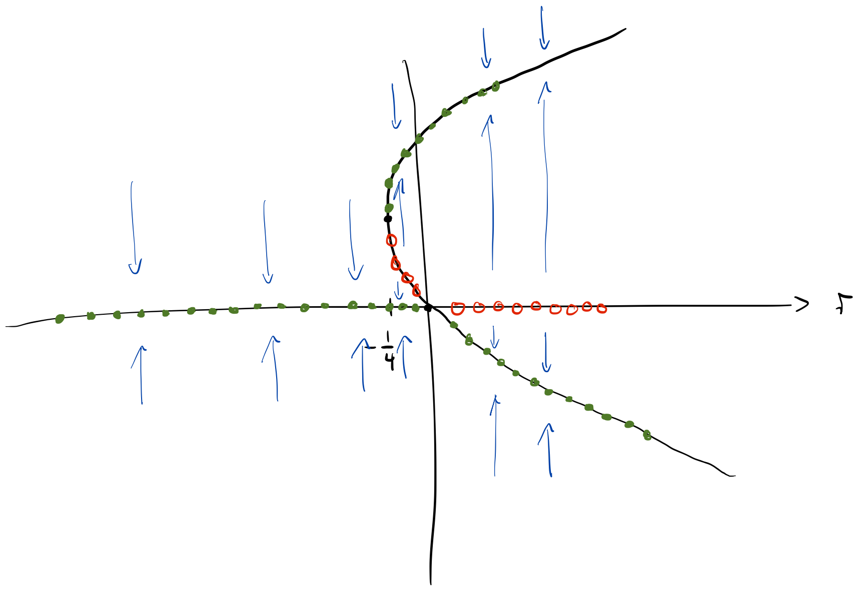
- So, for $r > -\frac{1}{4}$, we have two additional branches of equilibria given by $(*)$.

  $\Rightarrow$ saddle-node at $r = -\frac{1}{4}$

- The $\ominus$-branch of $(*)$ intersects $x=0$ if $\frac{1}{2} - \sqrt{\frac{1}{4}+r} = 0$, i.e. at $r=0$

  $\Rightarrow$ transcritical bifurcation at $r=0$

- $x=0$ is stable for $r<0$ and unstable for $r>0$. This is enough to draw the bifurcation diagram:



6. Consider a scalar differential equation of the form

$$\dot{x} = f(x)\,, \\ x(0) = x_0$$

and assume that $x(t) \leq c$ on some time interval $t \in [0, T]$. Suppose that $y(t)$ solves a modified equation with same initial condition,

$$\dot{y} = f(y) + \cos(\omega t)\,, \\ y(0) = x_0\,.$$

Show that $x(t)$ and $y(t)$ are close as $\omega \to \infty$, more precisely, that

$$z(t) = y(t) - x(t) = O(\omega^{-1})$$

for $t \in [0, T]$.

Hints: Derive an equation for $z^2(t)$ and integrate by parts. Further, recall the Gronwall inequality: If $C > 0$ and ϕ and ψ are continuous, non-negative, and satisfy

$$\phi(t) \leq \int_0^t \psi(s)\,\phi(s)\,\mathrm{d}s + C\,,$$

on $[0, T]$, then

$$\phi(t) \leq C\,\exp\left(\int_0^t \psi(s)\,\mathrm{d}s\right).$$

(5)

Following the hint:

$$\dot{z} = \dot{y} - \dot{x} = f(y) - f(x) + \cos \omega t$$

$$= \int_x^y f'(\xi)\,d\xi + \cos \omega t \qquad (*)$$

$$\Rightarrow \underbrace{z\dot{z}}_{= \frac{1}{2}\frac{d}{dt} z^2} = z \int_x^y f'(\xi)\,d\xi + z \cos \omega t$$

Integrating w.r.t. time:

$$\frac{1}{2} z^2(t) - \frac{1}{2}\underbrace{z^2(0)}_{=0} = \int_0^t z(s) \underbrace{\int_{x(s)}^{y(s)} f'(\xi)\,d\xi}_{\leq |y(s)-x(s)| \max_{|\xi| \leq 2c} |f'(\xi)| \equiv K|z(s)|}\,ds + \underbrace{\int_0^t z(s) \cos(\omega s)\,ds}$$

$$= z(s) \frac{1}{\omega} \sin(\omega s) \Big|_{s=0}^{s=t} - \frac{1}{\omega} \int_0^t \underbrace{\dot{z}(s)}_{\leq \text{bdd. by } (*)} \sin(\omega s)\,ds$$

(Solution ctd./scratch paper)

$$\Rightarrow \quad \frac{1}{2} z^2(t) \leq K \int_0^t z^2(s)\, ds + O\left(\frac{1}{\omega}\right)$$

So the claim follows by direct application of the Gronwall lemma.