# Differential Equations

Final Exam

July 22, 2024

1. Solve the scalar differential equation

$$\dot{x} - \frac{1}{t}\, x = \frac{t}{t+1} \quad \text{for } t > 0\,,$$
$$x(0) = 0\,.$$

(5)

2. Find the general solution to the two-dimensional system

$$\dot{x} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} x\,.$$

(10)

3. Consider the Volterra–Lotka system, here with all coefficients set to one,

$$\dot{x} = x - x\,y\,,$$
$$\dot{y} = x\,y - y\,.$$

(a) Find all equilibrium points of the system and determine their linear stability.

(b) Show that

$$V = x - \ln x + y - \ln y$$

is a Lyapunov function for the equilibrium point $(1, 1)$.

(c) Is $(1, 1)$ stable? Is it asymptotically stable? Explain!

(d) Sketch the phase portrait. If there are centers or foci, make sure that you get the orientation right. (5+5+5+5)

4. Consider the scalar equation with parameter $r$,

$$\dot{x} = r + x - \tfrac{1}{3}\, x^3 \,.$$

   (a) Find and classify all bifurcations.

   *Hint:* When looking for equilibria, it is easier to consider $r$ as a function of $x$ than the other way round.

   (b) Determine the stability of the equilibria and draw a bifurcation diagram.

(5+5)

5. Show that the following system has at least one periodic orbit:

$$\begin{aligned} \dot{x} &= x + y - x\,(x^2 + y^2)\,, \\ \dot{y} &= -x + y - y\,(x^2 + y^2)\,. \end{aligned}$$

(5)