1. Solve the scalar differential equation

$$\dot{x} - \frac{1}{t}\,x = \frac{t}{t+1} \quad \text{for } t > 0\,,$$
$$x(0) = 0\,.$$

(5)

Integrating factor $M = \exp\left(-\int_1^t \frac{ds}{s}\right) = e^{-\ln t} = \frac{1}{t}$

$\Rightarrow \frac{d}{dt}\left(\frac{x}{t}\right) = \frac{1}{t+1} \quad \Rightarrow \frac{x(t)}{t} = \ln(t+1) + C_1$

$\Rightarrow x(t) = t\ln(t+1) + C_1 t$

This function satisfies the initial condition for every value of C_1, so the solution is not unique. (RHS is not Lipschitz continuous in a neighborhood of $t=0$!)

2. Find the general solution to the two-dimensional system

$$\dot{x} = \underbrace{\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}}_{=:A} x \,.$$

(10)

Eigenvalues: $0 = \det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4 \quad \Rightarrow \quad \lambda_\pm = \pm 2$

Eigenvectors: $(A - 2I) v_+ = 0 \quad \Rightarrow \quad \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} v_+ = 0 \quad \Rightarrow \quad v_+ = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$(A + 2I) v_- = 0 \quad \Rightarrow \quad \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} v_- = 0 \quad \Rightarrow \quad v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \quad S = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

To compute S^{-1} use augmented matrix:

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 4 & 1 & -3 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & -\frac{3}{4} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{3}{4} \end{array}\right)$$

with $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix} = S^{-1}$

$\Rightarrow \quad A = S D S^{-1}$

General solution:

$$x(t) = e^{At} x_0 = S e^{Dt} S^{-1} x_0 = \frac{1}{4} \underbrace{\underbrace{\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}}_{= \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}}_{= \begin{pmatrix} 3e^{2t} + e^{-2t} & 3e^{2t} - 3e^{-2t} \\ e^{2t} - e^{-2t} & e^{2t} + 3e^{-2t} \end{pmatrix}} x_0$$

Remark: It is questionable whether the explicit form of the matrix is more helpful than

$x(t) = S e^{Dt} S^{-1} x_0$,

so full score for this form of the answer, too.

3. Consider the Volterra–Lotka system, here with all coefficients set to one,

$$\dot{x} = x - x\,y\,,$$
$$\dot{y} = x\,y - y\,.$$

(a) Find all equilibrium points of the system and determine their linear stability.

(b) Show that
$$V = x - \ln x + y - \ln y$$
is a Lyapunov function for the equilibrium point $(1,1)$.

(c) Is $(1,1)$ stable? Is it asymptotically stable? Explain!

(d) Sketch the phase portrait. If there are centers or foci, make sure that you get the orientation right. (5+5+5+5)

(a) $x - xy = 0 \quad \Rightarrow \quad x = 0$ or $y = 1$

$xy - y = 0 \quad \Rightarrow \quad y = 0$ or $x = 1$

$\Rightarrow$ equilibria at $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

@A: $Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow$ saddle, with e_1 unstable and e_2 stable directions.

@B: $Df = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix}_{\substack{x=1 \\ y=1}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $p(\lambda) = \lambda^2 + 1 \Rightarrow \lambda = \pm i$, linearly neutral

(b) Note that $g(x) = x - \ln x$ has $g'(x) = 1 - \frac{1}{x}$ which changes sign from $-$ to $+$ at $x = 1$

$\Rightarrow g$ has a global minimum at $x = 1$

$\Rightarrow V$ has a global minimum at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Set $\tilde{V}(x,y) = V(x,y) - V(1,1)$.

Then $\frac{d}{dt}\left(\tilde{V}(x(t), y(t))\right) = \nabla V \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left(1 - \frac{1}{x}\right)\dot{x} + \left(1 - \frac{1}{y}\right)\dot{y} = (x-1)(1-y) + (y-1)(x-1)$

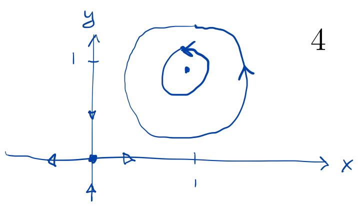
$= 0$

$\Rightarrow \tilde{V}$ is a Lyapunov function, but not a strict one.

(c) Existence of a Lyapunov function implies stability.

As orbits co-incide with level sets of V (or $\tilde{V}$), $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a center.

(d)



4. Consider the scalar equation with parameter r,

$$\dot{x} = r + x - \tfrac{1}{3}\,x^3\,.$$

(a) Find and classify all bifurcations.

Hint: When looking for equilibria, it is easier to consider r as a function of x than the other way round.

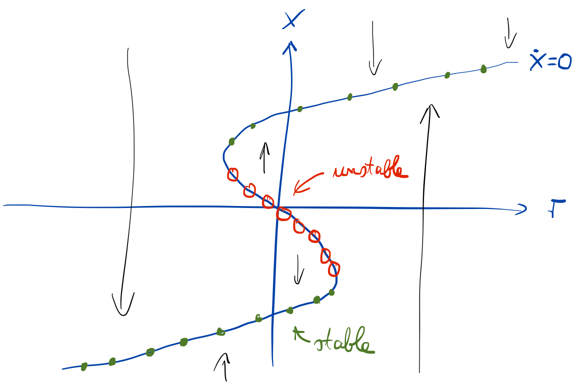
(b) Determine the stability of the equilibria and draw a bifurcation diagram.

(5+5)

(a) Equilibria lie on the curve $r = \frac{1}{3}x^3 - x$.

The number of equilibria as a function of r changes at the local extrema of this function. As $r' = x^2 - 1$, this happens when $x = \pm 1$, correspondingly $r = \mp \frac{2}{3}$. Both bifurcations are of saddle-node type.

(b)



To determine stability, note, for example, that $\dot{x} < 0$ for $x \gg 1$.

By continuity, this means $\dot{x} < 0$ in the upper left of the curve $\dot{x} = 0$

$\dot{x} > 0$ " " lower right " " " $\dot{x} = 0$.

The stability pattern shown is a direct consequence.

5. Show that the following system has at least one periodic orbit:

$$\dot{x} = x + y - x\,(x^2 + y^2)\,,$$
$$\dot{y} = -x + y - y\,(x^2 + y^2)\,.$$

(5)

Equilibria: $x + y - x(x^2+y^2) = 0 \Rightarrow x = 0$ or $x^2 + xy - x^2(x^2+y^2) = 0$

$-x + y - y(x^2+y^2) = 0 \Rightarrow y = 0$ or $-xy + y^2 - y^2(x^2+y^2) = 0$

$$x^2 + y^2 - (x^2+y^2)^2 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

But $x^2+y^2 = 1$ implies $x = y = 0$, a contradiction.

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only equilibrium point

Linearized system near the origin:

$$Df\Big|_{x=y=0} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{with } p(\lambda) = (1-\lambda)^2 + 1 = \lambda^2 - \lambda + 2$$

$$\Rightarrow \lambda_\pm = \frac{1}{2} \pm \sqrt{\underbrace{\tfrac{1}{4} - 2}_{<0}}$$

$\Rightarrow$ The origin is an unstable focus, and there exists a neighborhood U of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ that all orbits originating within leave.

Moreover, $x\dot{x} = x^2 + xy - x^2(x^2+y^2)$

$y\dot{y} = -xy + y^2 - y^2(x^2+y^2)$

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = x^2 + y^2 - (x^2+y^2)^2$$

Or, setting $V = x^2+y^2$: $\frac{1}{2}\dot{V} = V - V^2$

$\Rightarrow$ The ball $B(0,1)$ is positively invariant

Moreover, $B(0,1) \setminus U$ is pos. invariant and does not contain equilibria, so by Poincaré – Bendixon, it must contain a periodic orbit.