Recall:

$$G = \begin{pmatrix} 10 & -10 \\ -25 & 25 \end{pmatrix} \qquad \text{pay-off matrix}$$

mixed strategy for Robert $\quad r^T = (p, 1-p)$

Expected value for a fixed, given choice of Claire is

$$r^T G = \left(10p - 25(1-p),\ -10p + 25(1-p)\right)$$
$$= \left(35p - 25,\ -35p + 25\right)$$

Optimal strategy for Robert: get same expected payoff no matter what Claire plays.

$$35p - 25 = -35p + 25 \quad \Rightarrow 70p = 50 \quad \Rightarrow \quad p = \tfrac{5}{7}, \quad 1-p = \tfrac{2}{7}$$

With this choice of p, Robert's expected payoff is

$$35p - 25 = 35\,\tfrac{5}{7} - 25 = 0$$

From Claire's perspective, for a given row choice of Robert, the expected payoff is

$$G c = \begin{pmatrix} 10 & -10 \\ -25 & 25 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} 10q - 10(1-q) \\ -25q + 25(1-q) \end{pmatrix} = \begin{pmatrix} 20q - 10 \\ -50q + 25 \end{pmatrix}$$

optimal strategy is same expected payoff, no matter what Robert does:

$$20q - 10 = -50q + 25 \quad \Rightarrow \quad 70q = 35 \quad \Rightarrow \quad q = \tfrac{1}{2}$$
$$1-q = \tfrac{1}{2}$$

Claire's expected payoff: $20q - 10 = 20 \cdot \tfrac{1}{2} - 10 = 0$

"Value of the game is zero" — "the game is fair"

Another example:

$$G = \begin{pmatrix} \boxed{1} & -2^* \\ -3^* & \boxed{4} \end{pmatrix}$$

Is this game strictly determined? Row minima $*$ and column maxima $\square$ don't coincide $\Rightarrow$ no saddle point $\Rightarrow$ need mixed strategies.

Expected payoff for Robert:

$$(p \ , 1-p) \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} = (p - 3(1-p), -2p + 4(1-p)) = (4p-3, -6p+4)$$

Optimal strategy: $4p - 3 = -6p + 4 \Rightarrow 10p = 7 \quad p = \frac{7}{10}$ with expected payoff $4\frac{7}{10} - 3 = -\frac{2}{10}$

So value of the game (from Robert's perspective) is $-\frac{2}{10}$.

Expected payoff for Claire: $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} q - 2(1-q) \\ -3q + 4(1-q) \end{pmatrix} = \begin{pmatrix} 3q - 2 \\ -7q + 4 \end{pmatrix}$

Optimal strategy: $3q - 2 = -7q + 4 \Rightarrow 10q = 6 \Rightarrow q = \frac{6}{10}$ with expected payoff: $3\frac{6}{10} - 2 = -\frac{2}{10}$

Games with more than 2 choices:

Case 1: Reduction by dominance

$$G = \begin{pmatrix} -2^* & 6 & \boxed{4} \\ -1 & 2 & -3 \\ \boxed{1} & 2 & -2^* \end{pmatrix}$$

3rd row "dominates" 2nd row for Robert, then is no need to ever play second row.

3rd (and 1st) column dominate 2nd column for Claire, there is no reason to ever play second column.

$\Rightarrow$ reduced 2x2 payoff matrix, can find optimal strategies as before.

Need mixed strategy for Robert, optimal strategy has $p = \frac{1}{3}$.

Case 2: No reduction by dominance possible, no saddle point

From Robert's perspective:

- Use mixed strategy $r^T = (p_1, p_2, p_3, \dots p_n)$ if G has n rows.

  $p_1 + \dots + p_n = 1$, $p_i \geq 0$ for $i = 1, \dots, n$

- Wants to maximize expected payoff, independent of Claire's choice of column.

Mathematical description: "Linear programming problem" → Operations Research

maximize v

subject to $(r^T G)_j \geq v$ for $j = 1, \dots, m$ (Claire's choice)

"expected payoff given that Claire has chosen column j"

$p_1 + \dots + p_n = 1$, $p_i \geq 0$ for $i = 1, \dots, n$