Example: Sequence of coin tosses.

Goal: toss sequence HTH

Transition matrix

$$
\begin{array}{c|c}
 & \text{TO} \\
\text{FROM} &
\begin{array}{c|cccc}
 & HTH & HT & H & \text{empty} \\
\hline
HTH & 1 & 0 & 0 & 0 \\
HT & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
H & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\text{empty} & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
\end{array}
\qquad
\begin{array}{c|c}
I & 0 \\
\hline
R & Q
\end{array}
$$

Q: How many tosses will it take, on average, before getting HTH?

"mean exit time"

Recall: fundamental matrix $(I-Q)^{-1}$: entries are the average number of times the chain transitions from the state of the row index to the state of the column index.

$\Rightarrow$ row sums are the mean exit times from each given state.

Here: $I - Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Compute inverse:

$$
\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 \end{array}\right)
\xrightarrow{\frac{1}{2}R1 + R2 \to R2}
\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 \end{array}\right)
\xrightarrow[R2+R3 \to R3]{2R2 \to R2}
\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 2 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & 1 & 1 \end{array}\right)
$$

$$
\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 & 4 & 4 \end{array}\right)
\qquad \text{where } \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 4 & 4 \end{pmatrix} = (I-Q)^{-1}
$$

Row sums of $(I-Q)^{-1}$ are: $\begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}$

Interpretation:

From state HT it takes on average 6 tosses to exit

" " H " " " " 8 " " "

" " empty " " " " 10 " " "

Consistency check:

- From state HT: expected exit time is $1 + \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 10 = 1+5 = 6$ ✓

  $\frac{1}{2}\cdot 0$: complete on tossing H; $\frac{1}{2}\cdot 10$: back to empty on tossing T

- From state H: expected exit time is $1 + \frac{1}{2}\cdot 6 + \frac{1}{2}\cdot 8 = 1+3+4 = 8$ ✓

  $\frac{1}{2}\cdot 6$: to state HT on tossing T; $\frac{1}{2}\cdot 8$: to state H (last H can be kept) on tossing H

Remark: $(I-Q)^{-1}R = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 4 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ I.e.: from every initial state we exit with probability 1.

Game theory: Two-player zero-sum games

Example: Robert and Claire can decide to put a dime (10 ct) or a quarter (25 ct) in their hand. If sum $\geq$ 35 ct, Claire gets it all.

" " < 35 ct, Robert gets it all

Payoff matrix (Robert's payoff)

Robert's choice	Claire's choice: dime	Claire's choice: quarter
dime	10	-10
quarter	-25	-25

Robert's best choice: play a dime $\longrightarrow$ minimal loss independent of Claire's choice

Claire's best choice: play a quarter $\longrightarrow$ optimize her worst case

This is a <u>completely determined game</u>,

as for both players there is an optimal strategy that the other cannot "game"

How to detect a completely determined game:

- For every row, mark* minimum (worst case for a choice of the row player)
- For every column, mark$^{\square}$ maximum (" " " " " " column player)

If they coincide, this is a "saddle point", and determines the best strategy for both:

$$\begin{array}{ccc} & \multicolumn{2}{c}{C} \\ R & \boxed{10} & \boxed{-10}^* \\ & -25^* & -25^* \end{array}$$

saddle point solution,
so strategy is R: dime
C: quarter

with a deterministic loss of 10 ct per game for Robert.

Change setup: if two coins are the same, then Robert gets both
otherwise Claire gets both

$$\begin{array}{cccc} & & \multicolumn{2}{c}{C} \\ & & \text{dime} & \text{quarter} \\ R & \text{dime} & \boxed{10} & -10^* \\ & \text{quarter} & -25^* & \boxed{25} \end{array}$$

No saddle point,
i.e. no optimal deterministic strategy for any player

Guess: optimal random strategy for both:

play each choice randomly with $p = \frac{1}{2}$.