Continue with absorbing Markov chains.

- A state is <u>absorbing</u> if it is not possible to leave that state
- A chain is <u>absorbing</u> if (i) it has at least one absorbing state
  (ii) an absorbing state can be reached from every other state (in one or more steps)

In general, we can write the transition matrix for an absorbing chain as

$$\underbrace{\begin{array}{c|cc} & \multicolumn{2}{c}{\text{next}} \\ \text{current} & \text{abs} & \text{non-absorb.} \\ \hline \text{absor.} & I & O \\ \text{non-absorb.} & R & Q \end{array}}_{P}$$

Powers of the transition matrix

$$P^2 = \begin{pmatrix} I & O \\ R & Q \end{pmatrix}\begin{pmatrix} I & O \\ R & Q \end{pmatrix} = \begin{pmatrix} I\cdot I + O\cdot R & I\cdot O + O\cdot Q \\ RI + QR & R\cdot O + Q\cdot Q \end{pmatrix}$$

$$= \begin{pmatrix} I & O \\ R+QR & Q^2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} I & O \\ R+QR & Q^2 \end{pmatrix}\begin{pmatrix} I & O \\ R & Q \end{pmatrix} = \begin{pmatrix} I\cdot I + O\cdot R & I\cdot O + O\cdot Q \\ (R+QR)I + Q^2R & (R+QR)\cdot O + Q^2\cdot Q \end{pmatrix}$$

$$= \begin{pmatrix} I & O \\ R+QR+Q^2R & Q^3 \end{pmatrix}$$

$$P^k = \begin{pmatrix} I & O \\ \underbrace{(I+Q+Q^2+\dots+Q^{k-1})}_{A}R & Q^k \end{pmatrix}$$

<u>Fact</u>: For an absorbing chain,

$$\lim_{k\to\infty} Q^k = O$$

Q: How to effectively compute

$$A = (I + Q + Q^2 + Q^3 + \dots)R$$

A: Final transition matrix from non-absorbing to absorbing states.

Trick: $(I + Q + Q^2 + \dots + Q^{k-1})(I - Q)$

$$= I + \cancel{Q} + \cancel{Q^2} + \dots + \cancel{Q^{k-1}} - \cancel{Q} - \cancel{Q^2} - \dots - \cancel{Q^{k-1}} - Q^k$$

$$= I - \underbrace{Q^k}_{\to 0}$$

Let $k \to \infty$.

$$I + Q + Q^2 + \dots = (I - Q)^{-1}$$

$$\Rightarrow \quad A = (I - Q)^{-1} R$$

From calculus:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \qquad \text{Taylor series}$$

here:

$$(I - Q)^{-1} = I + Q + Q^2 + \dots$$

$A = (I - Q)^{-1} \cdot R$ $\qquad$ $A = (I + Q + Q^2 + \dots)R$

matrix multiplication

Let X_k be the random variable which is 1 if I transition from state i to state j in k steps, 0 otherwise.

$$\mathbb{E}[X_k] = Q^k$$

Then the expected number of times to be in state j if the initial state is i is

$$I + \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \dots = I + Q + Q^2 + \dots$$

$$= (I-Q)^{-1} =: N$$

What is the expected number of iterations before absorbtion?

$$N \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$ (row-sum of the entries of N)

Expected profit for the gambler's ruin problem:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ \vdots & \\ a_{41} & a_{42} \end{pmatrix}$$

Payoff $$B = \begin{pmatrix} -1000 & +4000 \\ -2000 & +3000 \\ -3000 & +2000 \\ -4000 & +1000 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ & \end{pmatrix}$$

$$\mathbb{E}[P] = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{12} \\ \vdots \\ a_{41} b_{41} + a_{42} b_{42} \end{pmatrix}$$