

## Markov chains:

- experiment whose outcomes form a discrete set (of states)
- Their probabilities depend on the current state, but do not depend on any of the past states.

If  $p_{ij}$  is the probability of transitioning from current state  $i$  to new state  $j$ , we can form the transition matrix

$$P = (p_{ij})$$

The entries of  $P$  in each row sum up to one.

### Example: Mating rabbits

Suppose each rabbit has a pair of genes that can occur in two forms  $g, G$ .

Then the rabbit can be

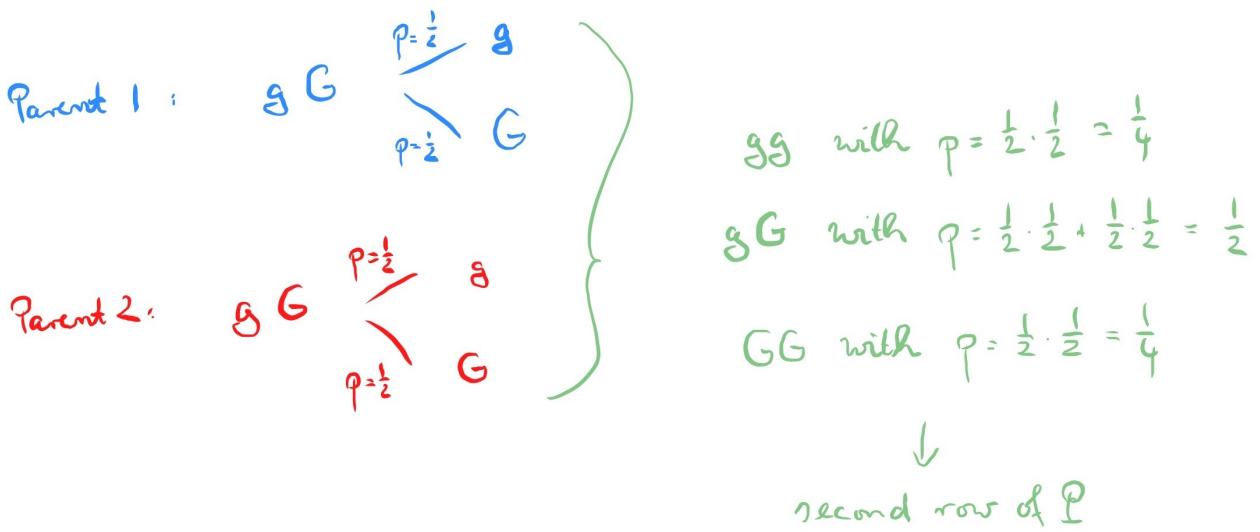
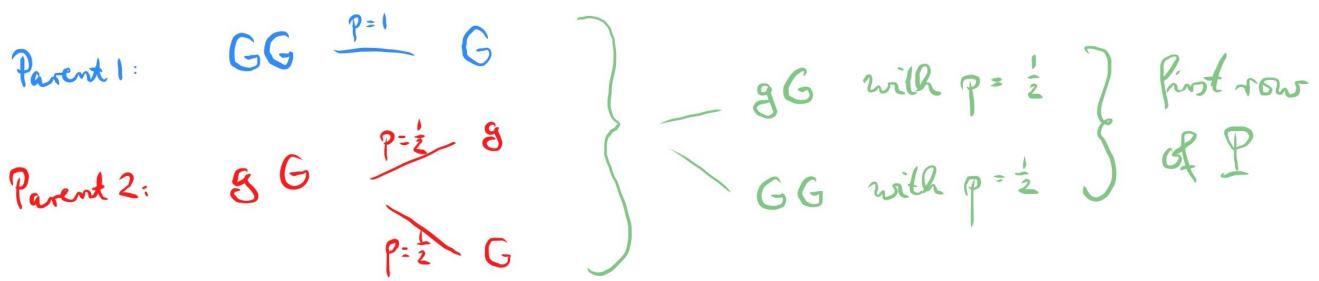
GG	gG = Gg	gg
"dominant"	"hybrid"	"recessive"

We look at the process of mating with a hybrid (or, equivalently, of mating randomly within a population where  $g$  and  $G$  occur with same frequency)

transition matrix:

	GG	gG	gg
GG	$\frac{1}{2}$	$\frac{1}{2}$	0
gG	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
gg	0	$\frac{1}{2}$	$\frac{1}{2}$
$\underbrace{\qquad\qquad\qquad}_{P}$			

Why?



Questions:

1. Is this chain regular?

(Recall that a chain is regular if  $P$  or any of its powers ( $P^2, P^3, \dots$ ) has only strictly positive entries)

Note that  $P$  has zero entries, but

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{8} & \frac{1}{4} + \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} + \frac{1}{8} & \frac{1}{8} + \frac{1}{4} + \frac{1}{8} & \frac{1}{8} + \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} + \frac{1}{4} & \frac{1}{8} + \frac{1}{4} \end{pmatrix}$$

has only strictly pos. entries  $\Rightarrow$  chain is regular.

2. What is the stationary distribution

$$x^T P = x^T \quad \text{or} \quad P^T x = x \quad \text{or} \quad (P^T - I)x = 0$$

$\Rightarrow$  Need to solve hom. linear system with matrix

$$\hat{P}^T - I = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \xrightarrow{\substack{-2R_1 \rightarrow R_1 \\ R_1 + R_2 \rightarrow R_2}} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \xrightarrow{\substack{R_1 + 2R_3 \rightarrow R_1 \\ -4R_2 \rightarrow R_2 \\ R_2 + R_3 \rightarrow R_3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for solution space is  $x = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

or, normalized as probability vector  $P = \frac{1}{x_1 + x_2 + x_3} x = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$

So, in the long run:

$\frac{1}{4}$  of rabbits are GG  
 $\frac{1}{2}$  " " " gG  
 $\frac{1}{4}$  " " " gg

### Absorbing Markov Chains

Gambler's ruin: Gambler in Vegas plays Black-Jack, prob. of winning is  $p = 0.4$

- will quit if has won \$5000, or is broke
- plays \$1000 per round

Transition matrix:

	OK	5K	1K	2K	3K	4K
OK	1	0	0	0	0	0
5K	0	1	0	0	0	0
1K	0.6	0	0	0.4	0	0
2K	0	0	0.6	0	0.4	0
3K	0	0	0	0.6	0	0.4
4K	0	0.4	0	0	0.6	0

Observe that

$$P = \begin{pmatrix} I & O \\ R & Q \end{pmatrix} \quad P^2 = \begin{pmatrix} I & O \\ R & Q \end{pmatrix} \begin{pmatrix} I & O \\ R & Q \end{pmatrix} = \begin{pmatrix} II + OR & IO + OQ \\ RI + QR & RO + Q^2 \end{pmatrix}$$
$$= \begin{pmatrix} I & O \\ R + QR & Q^2 \end{pmatrix}$$

If taking any power of  $P$ , the rows  $(I \ O)$  will never change

$\Rightarrow$  chain is not regular

Here: we ask: given some initial state, what is the probability of ending up in any of the absorbing states?