Markov chains:

- experiment whose outcomes from a discrete set (of states)
- Their probabilities depend on the current state, but do not depend on any of the past states.

If p_{ij} is the probability of transitioning from current state i to new state j, we can form the transition matrix

$$P = (p_{ij})$$

The entries of P in each row sum up to one.

Example: Mating rabbits

Suppose each rabbit has a pair of genes that can occur in two forms g, G.

Then the rabbit can be

GG	gG = Gg	gg
"dominant"	"hybrid"	"recessive"

We look at the process of mating with a hybrid (or, equivalently, of mating randomly within a population where g and G occur with same frequency)

transition matrix:

$$\begin{array}{c|ccc} & GG & gG & gg \\ \hline GG & \frac{1}{2} & \frac{1}{2} & 0 \\ gG & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ gg & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{P}$$

Child:

Why?

Parent 1: GG $\xrightarrow{p=1}$ G

Parent 2: gG $\xrightarrow{p=\frac{1}{2}}$ g, $\xrightarrow{p=\frac{1}{2}}$ G

gG with $p=\frac{1}{2}$

GG with $p=\frac{1}{2}$

First row of P

Parent 1: gG $\xrightarrow{p=\frac{1}{2}}$ g, $\xrightarrow{p=\frac{1}{2}}$ G

Parent 2: gG $\xrightarrow{p=\frac{1}{2}}$ g, $\xrightarrow{p=\frac{1}{2}}$ G

gg with $p=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

gG with $p=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

GG with $p=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

↓

second row of P

Questions:

1. Is this chain regular?

(Recall that a chain is regular if P or any of its powers ($P^2, P^3, \dots$) has only strictly positive entries)

Note that P has zero entries, but

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{8} & \frac{1}{4}+\frac{1}{4} & \frac{1}{8} \\ \frac{1}{8}+\frac{1}{8} & \frac{1}{8}+\frac{1}{4}+\frac{1}{8} & \frac{1}{8}+\frac{1}{8} \\ \frac{1}{8} & \frac{1}{4}+\frac{1}{4} & \frac{1}{8}+\frac{1}{4} \end{pmatrix}$$

has only strictly pos. entries $\Rightarrow$ chain is regular.

2. What is the stationary distribution

$$x^T P = x^T \quad \text{or} \quad P^T x = x \quad \text{or} \quad (P^T - I)x = 0$$

$\Rightarrow$ Need to solve hom. linear system with matrix

$$P^T - I = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R1+R2 \to R2]{-2R1 \to R1} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \xrightarrow[\substack{-4R2 \to R2 \\ R2+R3 \to R3}]{R1+2R3 \to R1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for solution pace is $x = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

or, normalized as probability vector $p = \dfrac{1}{x_1+x_2+x_3} x = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$

So, in the long run:

$\frac{1}{4}$ of rabbits are GG

$\frac{1}{2}$ " " " gG

$\frac{1}{4}$ " " " gg

Absorbing Markov chains

Gambler's ruin: Gambler in Vegas plays Black-Jack, prob. of winning is $p = 0.4$

- will quit of has won \$5000, or is broke
- plays \$1000 per round

Transition matrix:

$$\begin{array}{c|cc|cccc} & 0K & 5K & 1K & 2K & 3K & 4K \\ \hline 0K & 1 & 0 & 0 & 0 & 0 & 0 \\ 5K & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1K & 0.6 & 0 & 0 & 0.4 & 0 & 0 \\ 2K & 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 3K & 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 4K & 0 & 0.4 & 0 & 0 & 0.6 & 0 \end{array}$$

Observe that

$$P = \left(\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right) \qquad P^2 = \begin{pmatrix} I & O \\ R & Q \end{pmatrix}\begin{pmatrix} I & O \\ R & Q \end{pmatrix} = \begin{pmatrix} II + OR & IO + OQ \\ RI + QR & RO + Q^2 \end{pmatrix}$$

$$= \begin{pmatrix} I & O \\ R + QR & Q^2 \end{pmatrix}$$

If taking any power of P, the rows $(I \quad O)$ will never change

$\Rightarrow$ chain is not regular

<u>**Here:**</u> we ask: given some initial state, what is the probability of ending up in any of the absorbing states?