Recall: Random variable X is an assignment of a number x_i to each outcome that occurs with prob. p_i

Then the expected value of X is

$$E[X] = x_1 p_1 + x_2 p_2 + \dots + x_n p_n$$

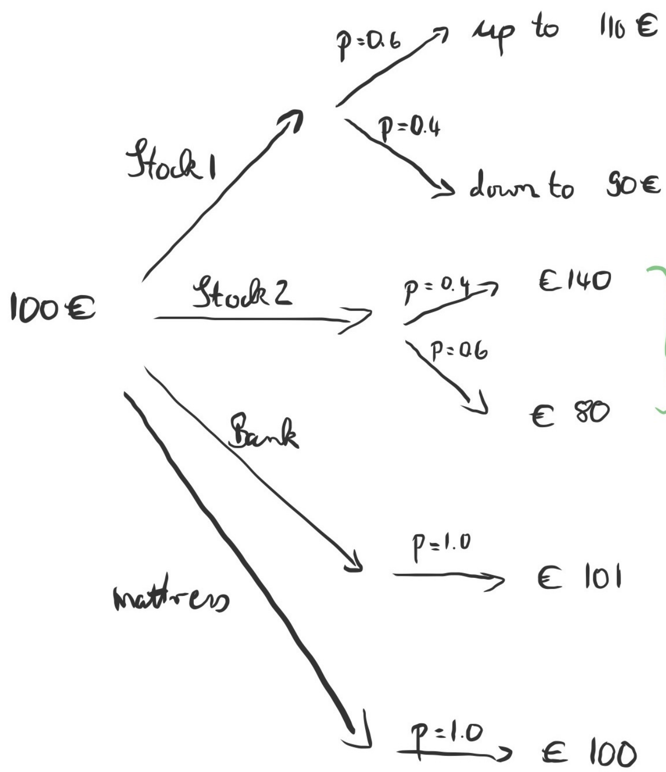
Examples: (3) Real estate broker spends 1200 € on advertising per house.

- If house sells in 3 months, gets 8000 €, 40% chance
- If not, listing is lost

Expected income (profit) per house:

$$E[P] = -1200 + 8000 \cdot \frac{4}{10} + 0 \cdot \frac{6}{10} = 3200 - 1200 = 2000$$

(4) Investment decisions based on risk-neutral valuation



$$E[P] = 110 \cdot \frac{6}{10} + 90 \cdot \frac{4}{10} = 102$$

$$E[P] = 140 \cdot \frac{4}{10} + 80 \cdot \frac{6}{10} = 104$$

chosen option, as highest expected value.

Remark: If the entire market is risk-neutral, then the core assumption in finance is that all expected profits are the same.

This means that one can infer risk-neutral probabilities as follows:

Suppose we start with 1 EUR, at end of the investment period, the risk-free profit ("Bank") is r EUR. Suppose the is a stock which can be up at u EUR, or down at d EUR. $d < r < u$

Require: $\mathbb{E}[P] = r$ "risk-neutral"

$$p u + (1-p) d = r$$ p: prob. that stock goes up

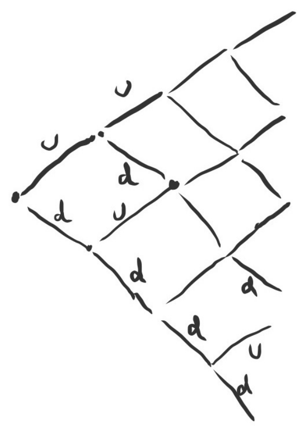
$$\Rightarrow p(u-d) = r - d$$

$$\boxed{p = \frac{r-d}{u-d}}$$

For Stock 1:

$$p = \frac{101-90}{110-90} = \frac{11}{20}$$

In practice, build larger tree of possible price movements.



Markov chains

Example:
- town served by two phone companies "Mama Bell" (M) and "Papa Bell" (P)
- both are trying to lure customers from the other
- each month: 40% of M- customers switch
  30% of P- " "

Transition matrix

$$\begin{array}{ccc} & & \text{next month} \\ & M & P \\ \text{this month} \quad \begin{matrix} M \\ P \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}} \end{array} =: P$$

Let $x_i = \begin{pmatrix} m_i \\ p_i \end{pmatrix}$ denote the number of customers for each in month i

$$x_{i+1}^T = x_i^T P \qquad\qquad x_{i+1} = P^T x_i$$

Q: Is there an equilibrium distribution of customers?

An equilibrium x will satisfy $x = P^T x$

$$\Rightarrow (P^T - I)x = 0$$

Here: $\underbrace{\left(\begin{pmatrix} \frac{6}{10} & \frac{3}{10} \\ \frac{4}{10} & \frac{7}{10} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)}_{A} x = 0$ hom. system of linear equations!

$$A = \begin{pmatrix} -\frac{4}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{3}{10} \end{pmatrix} \longrightarrow \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{pmatrix} \qquad \tilde{x} = \begin{pmatrix} -\frac{3}{4} \\ -1 \end{pmatrix}$$

$$x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

we can turn this into probabilities: $m = \frac{3}{7} = \frac{3}{3+4}$ $\quad m + p = 1$

$p = \frac{4}{7} = \frac{4}{3+4}$

1-D solution space of $Ax = 0$

Q: In general, does a unique equilibrium exist? Need A to be singular

If it exists, does any distribution of customers converge to the equilibrium distribution in the long run?

A Markov chain with transition matrix P is called <u>regular</u> if some power of P has only strictly positive entries.

For a regular Markov chain, the answer is <u>yes</u> to both questions.