Bayes' rule:

$$\underbrace{P(A|B)}_{\text{posterior}} = \frac{P(B|A)}{P(B)} \underbrace{P(A)}_{\text{prior}}$$

Example: ③ some machine is manufactured in 3 factories A, B, C

50% come from A, failure rate 6% $\Rightarrow P(F|A) = \frac{6}{100}$

30% " " B, " " 5% $\Rightarrow P(F|B) = \frac{5}{100}$

20% " " C, " " 4% $\Rightarrow P(F|C) = \frac{4}{100}$

Q: What is the prob. that a failing machine was made in A?

$$P(A|F) = \frac{P(F|A)\,P(A)}{P(F)}$$

$$P(F) = P(F|A)\,P(A) + P(F|B)\,P(B) + P(F|C)\,P(C)$$

$$= \frac{6}{100}\cdot\frac{5}{10} + \frac{5}{100}\cdot\frac{3}{10} + \frac{4}{100}\cdot\frac{2}{10}$$

$$= \frac{53}{1000}$$

$$\Rightarrow P(A|F) = \frac{\frac{30}{\cancel{1000}}}{\frac{53}{\cancel{1000}}} = \frac{30}{53} \approx 0.566$$

Review of ②:

Let D be the event that a person is using a drug

" T " " " " " " is testing positive for the drug

For the test: "sensitivity" $P(T|D) = \frac{97}{100}$ (true positives)

"specificity" $P(\bar{T}|\bar{D}) = \frac{95}{100}$ (true negatives)

Q: $P(D|T) = \frac{P(T|D)\,P(D)}{P(T)}$

also given: $P(D) = 0.5\% = \frac{1}{200}$

$$P(T) = P(T|D)\,P(D) + \overbrace{P(T|\bar{D})}^{1-P(\bar{T}|\bar{D})}\,\overbrace{P(\bar{D})}^{1-P(D)} = \frac{97}{100}\cdot\frac{1}{200} + \frac{5}{100}\cdot\frac{199}{200} = \frac{1092}{20000} \approx 0.05$$

Bernoulli trials

- only two outcomes, success and failure
- repeated n times
- outcomes are independent of the previous ones, with $P(\text{success}) = p$
  $P(\text{failure}) = 1 - p = q$

Q: $P(k \text{ successes in } n \text{ trials}) = P(n, k; p)$ (*)

Simpler problem:

$$P(k \text{ successes followed by } n-k \text{ failures}) = \underbrace{p \cdot p \cdots p}_{k \text{ times}} \cdot \underbrace{q \cdots\cdots q}_{n-k \text{ factors}} = p^k q^{n-k}$$

For (*), need to multiply by all the ways k successes and $n-k$ failures can be arranged: C^n_k

$$\Rightarrow \boxed{P(n,k;p) = C^n_k \cdot p^k q^{n-k}, \qquad q = 1-p}$$

Examples: 1 coin is flipped 10 times, what is the prob. of getting heads 3 times?

$$P(10, 3; \tfrac{1}{2}) = C^{10}_3 \cdot \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^{10-3}$$

$$C^{10}_3 = \frac{10!}{3!\ 7!}$$

$$= \frac{\cancel{8}\cdot \cancel{9}^{3} \cdot \cancel{10}^{5}}{1 \cdot \cancel{2} \cdot \cancel{3}} \left(\tfrac{1}{2}\right)^{\cancel{10}\ 7}$$

$$= \frac{15}{128} \approx 0.12$$

2 Medicine cures 80% of patients

What is the prob. that out of 8 people, 5 will be cured.

$$P(8, 5; \tfrac{8}{10}) = \frac{8!}{5!\ 3!} \left(\tfrac{8}{10}\right)^5 \left(\tfrac{2}{10}\right)^3 \approx 0.15$$

Expected value

Example:
- roll a die
- If 1, ..., 5 is shown, opponent will pay you face value of die (in EUR)
- If 6 is shown, you have to pay 18 EUR

Should you want to play this game?

Expected payout: $1 \cdot \frac{1}{6} + 2 \frac{1}{6} + 3 \frac{1}{6} + 4 \frac{1}{6} + 5 \cdot \frac{1}{6} - 18 \frac{1}{6}$

$= \frac{15}{6} - \frac{18}{6} = -\frac{1}{2}$

$\Rightarrow$ On average, you lose 50 ct. per round.

General principle:

A random variable X is a number associated with each possible outcome of an experiment. (For mathematicians: $X: S \to \mathbb{R}$)

If there are a finite number of outcomes, each associated with value $X = x_i$, occurring with probability p_i, then the expected value is defined

$$E[X] = x_1 p_1 + x_2 p_2 + \dots + x_n p_n$$

Further examples:

① Let C be the number of children per family. In some town, 10% of families have 3 children, 60% have 2, 20% have 1, 10% have 0.

Q: average number of children per family?

$$\mathbb{E}[C] = 3 \cdot \frac{1}{10} + 2 \cdot \frac{6}{10} + 1 \frac{2}{10} + 0 \cdot \frac{1}{10} = \frac{17}{10} = 1.7$$

③ Lottery: choose 6 numbers out of 51

6 matching numbers give 2 000 000 EUR, a ticket costs 1 EUR.

Expected payoff: $-1 + 2000\,000 \cdot \dfrac{1}{C^{51}_{6}} \approx -0.89$