Recall: $P(E) = \dfrac{|E|}{|S|}$

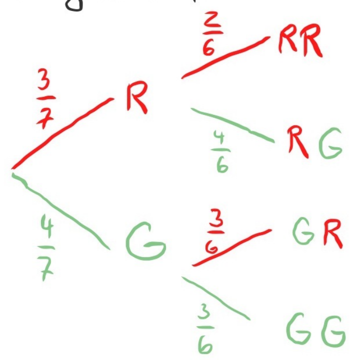
Example: Jar contains 3 red and 4 green marbles, two are drawn

Q: what is the probability that both marbles are red?

(i) with replacement: $p = \frac{3}{7} \cdot \frac{3}{7} = \frac{9}{49}$ where $\frac{3}{7}$ is the prob. of the event of drawing a red marble from the set of 7

(ii) without replacement: $p = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$ (when the first red marble is drawn, there are two red out of 6 left)

Tree diagram for (ii)



$p = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$

$p = \frac{3}{7} \cdot \frac{4}{6} = \frac{2}{7}$

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$p = \frac{4}{7} \cdot \frac{3}{6} = \frac{2}{7}$

We can now, e.g., read off that

$P(\text{drawing 1 red, 1 green}) = \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$

$P(\text{drawing zero red}) = \frac{2}{7}$

...

Solving the same problem via counting principles:

$$P(\text{drawing 2 red}) = \frac{\#\text{ possibilities of drawing two red out of 3}}{\#\text{ possibilities of drawing two out of 7}}$$

$$= \frac{C^3_2}{C^7_2}$$

$$= \frac{\frac{3!}{1!\,\cancel{2!}}}{\frac{7!}{5!\,\cancel{2!}}}$$

$$= \frac{3 \cdot 2}{7 \cdot 6} = \frac{1}{7}$$ (as before!)

Example: 5 cards are drawn from a standard deck

Q: P(getting two pairs)

In how many ways can this be done?

1. Select 3 values out of 13

$$C^{13}_3$$

2. Select the value which is not a pair

$$3 = C^3_2 = C^3_1$$

3. Choose the suits for the cards from pair 1 : C^4_2
4. " " " " " 2 : C^4_2
5. Choose the suit for the single card : C^4_1

2 → 2 clubs, 2 diamonds

3

⋮

value → 3 spades, 3 hearts

⋮

10 → 10 hearts

J

Q

K

Ace

suits (4)

A:
$$\frac{C^{13}_3 \, C^3_1 \, C^4_2 \, C^4_2 \, C^4_1}{C^{52}_5} = \frac{198}{4165}$$

Conditional probability

Example: A family has 3 children, what is the prob. that 2 children are boys given that the oldest is a boy?

$$S = \{GGG, GGB, GBG, \underline{GBB}, BGG, \underline{BGB}, \underline{BBG}, BBB\}$$

E

A: $p = \frac{2}{4}$

$$P(B \text{ given } E) = \frac{|B \cap E|}{|E|} = \frac{2}{4}$$

E : oldest is a boy :

B : $B = \{\cancel{GBB}, BGB, BBG\}$

$$P(B \mid E) = \frac{\frac{|B \cap E|}{|S|}}{\frac{|E|}{|S|}} = \frac{P(B \cap E)}{P(E)}$$

"prob. of B given E"

The conditional probability "B given E"

$$\boxed{P(B|E) = \frac{P(B \cap E)}{P(E)}}$$

Example: People commuting to work

	M	F	
drive D	8	13	21
public trans. P	39	40	
	47		

$P(D|M) = \frac{8}{47}$

$P(F|D) = \frac{13}{21}$

Two events E, F are independent if

$$P(E) \overset{(*)}{=} P(E|F)$$

then $P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{\cancel{P(E|F)}\, P(F)}{\cancel{P(E)}} = P(F)$

Example:

	M	F
color-blind C	6	1
not color-blind N	46	47

Q: Is color-blindness independent of gender?

$P(C) = \frac{7}{100}$ $100 = 6+1+46+47$

$P(C|M) = \frac{6}{52} > \frac{7}{100}$

$\Rightarrow$ color-blindness is not independent of gender.

Another example:

In a deck of cards, is value independent of suit?

$$P(K) = \frac{4}{52} = \frac{1}{13}$$

$$P(K \mid \text{hearts}) = \frac{1}{13}$$

$\Rightarrow$ drawing a king is independent from drawing hearts.

one king of hearts

13 hearts of all values