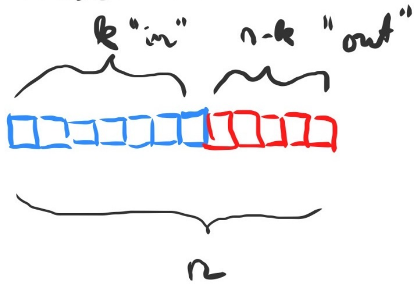
Combinations

Example: In how many ways can a two-person committe be formed from among {Alice, Bob, Claire}

A: $\{A,B\}$, $\{A,C\}$, $\{B,C\}$, so number is 3.

In general: Need the number of ways that a k-element subset can be chosen from an n-element set.



$$C_k^n = \frac{n!}{k!\,(n-k)!}$$

# of arrangements in order

# of re-arrangement of the "in"-positions

# of re-arrangement of the "out"-positions

Example: 1. An exam paper says: Answer 5 out of 7 questions.

In how many ways can this be done?

$$C_5^7 = \frac{7!}{5!\,\underbrace{(7-5)!}_{2! = 2\cdot 1}} = \frac{7\cdot 6}{2\cdot 1} = 21$$

$$\frac{7!}{5!} = \frac{7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

Even more generally: n distinct elements can be partitioned into $r_1, r_2, \ldots, r_k$ different subsets in

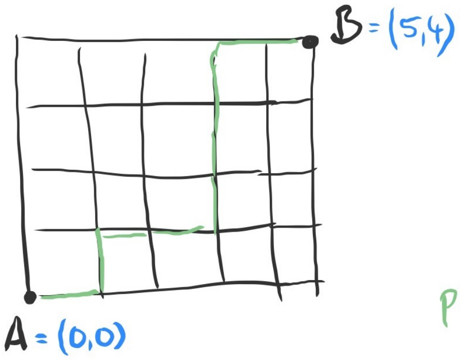
$$\frac{n!}{r_1!\cdot r_2! \cdots r_k!}$$

ways. $r_1 + r_2 + \ldots + r_k = n$

Compare

$$C_k^n = \frac{n!}{k!\,(n-k)!}$$

Example:



In how many different shortest paths can we travel from A to B?

path: RURRUUURR

Solution: 9 road segments, 4 up, 5 right

$$A: \ C_4^9 = C_5^9 = \frac{9!}{4! \ 5!}$$

We can also think about this as permutations: 9 segments, 4U, 5R are to put into a final order.

$$\frac{9!}{4! \ 5!} = \frac{9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6}^{2}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 9 \cdot 14 = 126$$

3. In how many ways can a committee of 5 be chosen from among 4 men and 4 women, if there are no more than 3 of any gender on the committee?

Answer: either 3M, 2W : $C_3^4 \cdot C_2^4 = \frac{4!}{3! \ 1!} \ \frac{4!}{2! \ 2!} = 4 \cdot 3! = 24$

or 2M, 3W : $C_2^4 \cdot C_3^4 = 24$

48 possibilities in total.

Probability

Ex.:

rolling a die

trial: the act of performing an experiment which can be repeated under same conditions an arbitrary number of times

outcome: result of a trial

1, 2, 3, 4, 5 or 6

sample space: the set of all possible outcomes

$\{1,2,3,4,5,6\} = S$

events: subsets of the sample space (collections of outcomes)

$E = \{2,4,6\}$ "event of rolling an even number"

$T = \{3,4,5,6\}$ "event of rolling at least 3"

probability: $P(A)$ is the relative number of occurrance of A in a large number of trials.

$$P(A) = \frac{|A|}{|S|} = \frac{\#\text{ elem. of } A}{\#\text{ elem. in } S}$$

usually, assume that all outcomes are equally likely ("simple events")

Example: Jar contains marbles numbered 1, 2, 3

2 are drawn

What is the prob. of the event A that their sum is at least 4?

$$S = \{\underbrace{(1,2)}_{3}, \underbrace{(2,3)}_{5}, \underbrace{(1,3)}_{4}\}$$

$$A = \{(2,3), (1,3)\}$$

$$P(A) = \frac{|A|}{|S|} = \frac{2}{3}$$

$$13\left\{\begin{array}{l} 2 \\ 3 \\ 4 \\ \vdots \\ 10 \\ J \\ Q \\ K \\ \text{Ace} \end{array}\right.$$

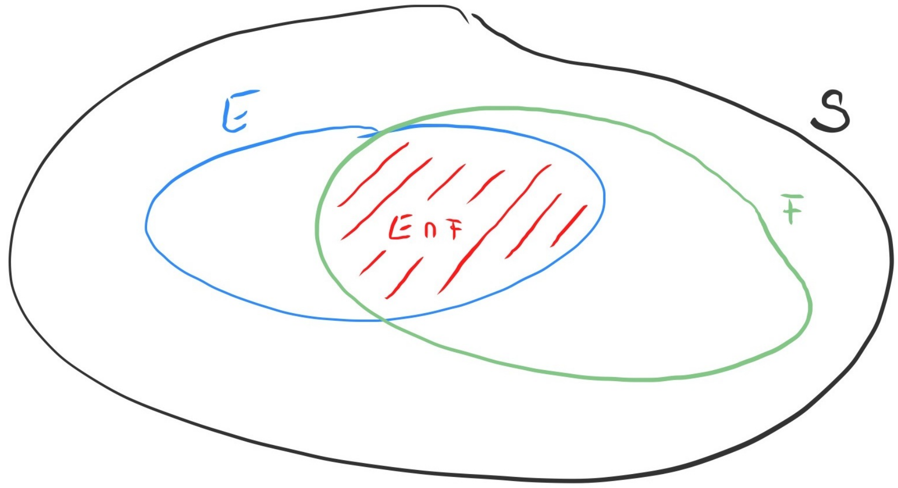
$\underbrace{\text{diamonds, hearts, club, spades}}_{4 \text{ suits}}$

52 cards in total

Def. events E, F are <u>mutually disjoint</u> if $E \cap F = \emptyset$

Ex.: $S = \{\text{card drawn from a standard deck of cards}\}$

$E = \{\text{card is an ace}\}$, $F = \{\text{card is hearts}\}$, $E \cap F = \{\text{ace of hearts}\} \Rightarrow$ not mut. disjoint



$$|E \cup F| = |E| + |F| - |E \cap F|$$

$$\Rightarrow \boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If E, F are mutually disjoint, then $P(E \cup F) = P(E) + P(F)$

In particular: $P(\overline{E}) = 1 - P(E)$ because: $P(S) = 1$, E and $\overline{E}$ are m. disjoint

Previous example:

$$P(S) = \frac{|S|}{|S|} = 1$$

$P(E)$ = prob. of drawing an ace $= \frac{4}{52}$

$P(F)$ = prob. of " hearts $= \frac{13}{52}$

$P(E \cup F)$ = prob. of drawing an ace or hearts $= \frac{4}{52} + \frac{13}{52} - \underbrace{P(E \cap F)}_{\frac{1}{52}} = \frac{16}{52}$

$= \frac{4}{13}$

Recall that $E \cap F$
$= \{$ace of hearts$\}$

<u>Another example:</u> Two dice are rolled

$F = \{$sum is $4\}$

$T = \{$at least one dice shows $3\}$

$F = \{(1,3), (2,2), (3,1)\}$ $\quad |F| = 3$

$T = \{\underbrace{(3,1), (3,2), (3,3), \ldots (3,6)}_{6}, \underbrace{(1,3), (2,3), (4,3), (5,3), (6,3)}_{5}\}$ $\quad |T| = 11$

$|F \cap T| = 2$

$P(\text{sum is 4 or at least one die is 3}) = P(F \cup T)$

$$= \frac{3}{36} + \frac{11}{36} - \frac{2}{36} = \frac{12}{36} = \frac{1}{3}$$

$36 = |S|$

$= 6 \cdot 6$

# of different faces of die #1

# of different faces of die #2