1. Compute the distance of the point $\begin{pmatrix}0\\1\\0\end{pmatrix}$ to the line through the origin in the direction of the vector $v = \begin{pmatrix}-1\\1\\1\end{pmatrix}$.

Interpret as: "solve" $\underbrace{\begin{pmatrix}-1\\1\\1\end{pmatrix}}_{A}\underbrace{\lambda}_{x} = \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{b}$

$AA^T \in Mat(3\times 3)$

$\underbrace{A^TA} \in Mat(1\times 1)$

$(-1\ 1\ 1)\begin{pmatrix}-1\\1\\1\end{pmatrix} = 1+1+1 = 3$

$A^Tb = (-1\ 1\ 1)\begin{pmatrix}0\\1\\0\end{pmatrix} = 1$

Recall: Least-square solution is given by

$$x = (\underbrace{A^TA}_{=3})^{-1}\underbrace{A^Tb}_{=1} = \frac{1}{3}$$

$\Rightarrow$ closest point on the line is $\begin{pmatrix}-1\\1\\1\end{pmatrix}\frac{1}{3}$

To compute the actual distance, take

$$\left\| \frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} \right\| = \left\| \begin{pmatrix}-1/3\\-2/3\\1/3\end{pmatrix} \right\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}} = \frac{\sqrt{6}}{3}$$

In general, a line is given by

$$x = a + \lambda v$$

direction along the line

point on the line

Take a column vector $x \in \mathbb{R}^n$, $n \geq 2$

Claim is $x x^T$

is not invertible

$$\begin{pmatrix} x_1 x_1 & x_1 x_2 & \\ x_2 x_1 & x_2 x_2 & \cdots \\ x_3 x_1 & x_3 x_2 & \\ \vdots & \vdots & \end{pmatrix} = B$$

all columns are multiples of each other

$\Rightarrow$ rank $B = 1$

<u>HW5 #1:</u> A has l.i. columns, least-square solution of $Ax = b$ is given by $x = (A^T A)^{-1} A^T b$

$$x = (A^T A)^{-1} A^T b$$

$$\Leftrightarrow \quad A^T A x = A^T b$$

$$\Leftrightarrow \quad A^T A x - A^T b = 0$$

$$\Leftrightarrow \quad A^T (Ax - b) = 0$$

$$A^T A x = \underbrace{A^T A (A^T A)^{-1}}_{= I} A^T b$$

$$A = \begin{pmatrix} | & | & \\ a_1 & a_2 & \cdots \\ | & | & \end{pmatrix} \quad \text{"tall"}$$

For each of the columns of A:

$$a_i^T \underbrace{(Ax - b)}_{\text{residual } r} = 0$$

"residual is perpendicular to columns of A"

r is a vector s.t.

$A^T r = 0$

$Ax - r = b$

This was to be shown $(y = -r)$

Easy solution (backward):

Given $A^T y = 0$ (**)

$Ax + y = b$ (*)

Then multiply (*) by A^T from the left:

$$A^T A x + \underbrace{\cancel{A^T y}}_{=0 \text{ due to } (**)} = A^T b$$

$$\Rightarrow A^T A x = A^T b$$

$$\Rightarrow x = (A^T A)^{-1} A^T b$$

#2

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{b} \Rightarrow A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b \Leftrightarrow A^T A x = A^T b$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right) \Rightarrow \boxed{x = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}}$$

#3: $y = b \cdot 1 + mx$

We need to solve in the least-square sense:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} b \\ m \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}}_{y}$$

$(0,3)$
$(1,3)$
$(1,6)$

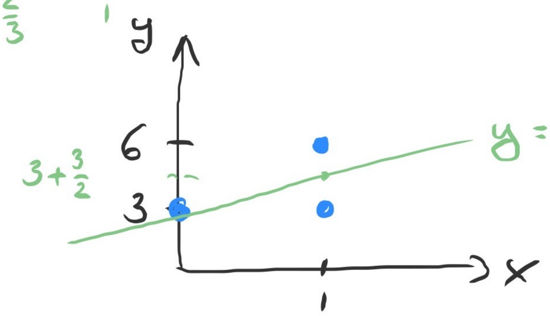
$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$$

$$A^TAx = A^Tb$$

$$\left(\begin{array}{cc|c} 3 & 2 & 12 \\ 2 & 2 & 9 \end{array}\right) \xrightarrow[-\frac{2}{3}R1 + R2 \to R2]{\frac{1}{3}R1 \to R1} \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 4 \\ 0 & \underbrace{2 - \frac{4}{3}}_{\frac{2}{3}} & \underbrace{9 - 8}_{1} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 4 \\ 0 & 1 & \frac{3}{2} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 4-1 \\ 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} \end{pmatrix}$$



Fitting exponential growth to Corona-epidemic data:

$$I = c \cdot e^{rt}$$

t: time in days

Take the log:

$$\ln I = \ln c + r t$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \end{pmatrix}}_{A} \begin{pmatrix} \ln c \\ r \end{pmatrix} = \begin{pmatrix} \ln I_1 \\ \ln I_2 \\ \vdots \end{pmatrix}$$