# Review problems Mo, March 16

1. Are the following maps linear?

(a) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

Answer: yes, we can write it as matrix times vector,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(Note: any map of the form $F(v) = Av$ is linear, as

$$F(v+w) = A(v+w) = Av + Aw = F(v) + F(w)$$

and $F(\lambda v) = A(\lambda v) = \lambda Av = \lambda F(v)$ )

(b) The shift by one unit in $y$-direction in the plane:

Answer: No, since

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{offset}}$$

so $F$ is affine, but not linear.

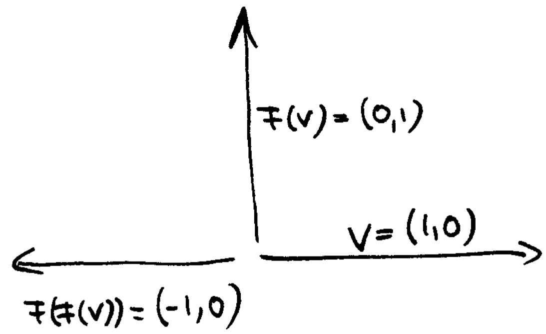
(c) $F$ denotes the number of non-zero entries of a vector $v$.

Answer: No, since

$$F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \qquad F\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 1 \neq 2 \cdot F\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(d) The rotation in the plane by 90° (anti-clockwise):

Yes, as $\quad F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$




Note: In general, we can check the angle of rotation of

$$w = F(v)$$

using the formula

$$\cos\alpha = \frac{w^T v}{\|w\| \, \|v\|}$$

Here, $v = \begin{pmatrix} x \\ y \end{pmatrix} \qquad w = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

so that $\quad v^T w = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} = -xy + yx = 0$

$\Rightarrow \cos\alpha = 0 \quad \Rightarrow \quad \alpha = 90°$ (for orientation, see sketch above)

2. Let

$$A = \begin{pmatrix} 4 & 4 & 1 & 7 \\ 3 & 3 & 0 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}$$

(a) Find the general solution to $Ax = b$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 4 & 4 & 1 & 7 & -5 \\ 3 & 3 & 0 & 6 & -6 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{-4R3+R1 \to R2 \\ -3R3+R1 \to R3}]{R3 \to R1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 3 & -9 \\ 0 & 0 & -3 & 3 & -9 \end{array}\right)$$

$$\xrightarrow[\frac{R2}{-3} \to R2]{\text{remove } R3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R1 - R2 \to R1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) (*)$$

$$\text{So } x = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{\lambda \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}}_{\text{general solution to } Ax=0}$$

Check: $$A \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 1 & 7 \\ 3 & 3 & 0 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix} \checkmark$$

$$A \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark \qquad A \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

(b) Find the least-norm solution to $Ax=b$

Recall from class: If $A$ has rows that are l.i., then

$$x = A^T (AA^T)^{-1} b$$

<u>But</u>: Here there rows of $A$ are l.d., as we see from the previous calculation that rank $A=2$. (not 3!)

<u>Solution</u>: replace $A, b$ by matrix/vector from further down the Gauss elimination process once it is clear that the remaining columns are l.i.. For example

$$Bx = c$$

with

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

from (*).

So the least-norm solution is

$$x = B^T \underbrace{(BB^T)^{-1} c}_{=:y}$$

so that $y = (BB^T)^{-1} c$

or $BB^T y = c$ (**)

$$BB^T = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$$

To solve (**), write augmented matrix

$$\left(\begin{array}{cc|c} 6 & -2 & -2 \\ -2 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -\frac{3}{2} \\ 0 & 4 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -\frac{3}{2} \\ 0 & 1 & \frac{7}{4} \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{7}{4} \end{array}\right)$$

so that $y = \begin{pmatrix} \frac{1}{4} \\ \frac{7}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

and $x = B^T y = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 7 \\ -5 \end{pmatrix}$

<u>Check:</u> $Ax = \frac{1}{4}\begin{pmatrix} 4 & 4 & 1 & 7 \\ 3 & 3 & 0 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 7 \\ -5 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 15-35 \\ 6-30 \\ 9-5 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}$ ✓

For least-norm solution, $\|x\|^2 = \frac{1}{16}(1+1+49+25) = \frac{76}{16} = \frac{19}{4}$

Particular solution from (a), in comparison, has norm

$$\left\| \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\|^2 = 4+9 = 13 > \frac{19}{4}.$$