Mock Midterm Solutions

1. Need two vectors in the plane, e.g.:

$$v = a - b = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \qquad w = a - c = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

So parametric description is

$$x = a + \lambda v + \mu w = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

2. (a) True. (That's the whole point of doing elementary row transformations.)

(b) Sometimes true: when the constant is nonzero.

(c) False. (If the solution is not unique, there are infinitely many.)

(d) True.

(e) False. ($x=0$ is always a solution of $Ax=0$.)

(f) Sometimes true. (If A has l.i. columns, then $Ax=0 \Rightarrow x=0$.)

(g) Sometimes true. (Only if system is consistent)

(h) False. (rank $A \leq 4$, so A cannot have more than 4 l.i. columns.)

(i) Sometimes true. (True if and only if the columns of A are l.i.)

(j) True.

3 (a).
$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 1 \\ 2 & 6 & 0 & -4 & 1 \end{array}\right) \xrightarrow{-2R1+R2 \to R2} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 1 \\ 0 & 0 & 0 & 4 & -1 \end{array}\right)$$

$$\xrightarrow[R2+R1 \to R1]{\frac{1}{4}R2 \to R2} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} \end{array}\right)$$

So $$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{4} \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

(b) $$AA^T = \begin{pmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 0 & 0 \\ -4 & -4 \end{pmatrix} = \begin{pmatrix} 26 & 36 \\ 36 & 56 \end{pmatrix}$$

$$x = A^T \underbrace{(AA^T)^{-1} b}_{y} \quad \Rightarrow \quad AA^T y = b \qquad (*)$$

Solve $(*)$:
$$\left(\begin{array}{cc|c} 26 & 36 & 1 \\ 36 & 56 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{18}{13} & \frac{1}{26} \\ 0 & \frac{80}{13} & -\frac{5}{13} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{18}{13} & \frac{1}{26} \\ 0 & 1 & -\frac{1}{16} \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{8} \\ 0 & 1 & -\frac{1}{16} \end{array}\right) \qquad \Rightarrow y = \frac{1}{16} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 0 & 0 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \frac{1}{16} = \frac{1}{16} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{4} \end{pmatrix}$$

Alternative solution for (b):

In this particular case, we notice that the general solution from (a) is of the form

$$x = a + \lambda u + \mu v$$

where a, u, and v are mutually orthogonal. Then, by the Pythagorean theorem,

$$\|x\|^2 = \|a\|^2 + \|\lambda u\|^2 + \|\mu v\|^2$$
$$= \|a\|^2 + \lambda^2 \|u\|^2 + \mu^2 \|v\|^2 .$$

So clearly $\|x\|^2$ is minimal if and only if $\lambda = \mu = 0$.

(A variation of this approach is to write out $\|x\|$ explicitly and note that the resulting expression is minimized when $\lambda = \mu = 0$.)

4. $u^T v = -1 + 0 + 1 = 0 \Rightarrow u, v$ perp.

$v^T w = 0 + 0 + 0 = 0 \Rightarrow v, w$ perp.

$u^T w = 0 + 12 + 0 \neq 0 \Rightarrow u, w$ not perp.

5. Want to solve

$$p \text{ (or } q) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

For p:

$$\underbrace{\begin{pmatrix} -5 \\ -1 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$$

In matrix-form:

$$\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$$

augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 3 & -4 \\ -1 & 0 & -2 \\ 2 & -1 & 6 \end{array}\right) \xrightarrow[2R2+R3 \to R3]{R1+R2 \to R2} \left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 3 & -6 \\ 0 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[-R3 \to R3]{3R3+R2 \to R2} \left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right) \xrightarrow{-3R3+R1 \to R1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right)$$

$$\Rightarrow \lambda = 2, \quad \mu = -2$$

p is in the plane.

Suspecting that then q is not in the plane, we go for the least-square solution right away (we can check later!):

Here, $b = \begin{pmatrix} 3 \\ 32 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 31 \\ 5 \end{pmatrix}$

$$\Rightarrow A^T b = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 31 \\ 5 \end{pmatrix} = \begin{pmatrix} 4-31+10 \\ 12+0-5 \end{pmatrix} = \begin{pmatrix} -17 \\ 7 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 10 \end{pmatrix}$$

Thus, to solve $A^TAx = A^Tb$, we use the augmented matrix

$$\left(\begin{array}{cc|c} 6 & 1 & -17 \\ 1 & 10 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 10 & 7 \\ 0 & -59 & -59 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 10 & 7 \\ 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right)$$

So the least-square solution has $\lambda = -3$, $\mu = 1$, and the point on the plane closest to q is

$$x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} \neq q$$

So q is not a point in the given plane.

6. We need to show that $A^TAx = 0$ implies $x = 0$.

So suppose $A^TAx = 0$

$$\Rightarrow x^TA^TAx = 0$$

$$\Rightarrow \|Ax\|^2 = 0$$

$$\Rightarrow Ax = 0$$

Since the columns of A are l.i., this implies $x = 0$.