Finite Mathematics

Makeup Final Exam

August 25, 2020, 17:00–19:00

Last Name: ____

First Name: ____

Signature: ____

1. Are the following vectors linearly independent? If not, determine a maximal set of linearly independent vectors, and express the other vectors as linear combinations of vectors from this set.

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}_4 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix}.$$

(10)

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -3 & 3 \\ 0 & -1 & 1 & -2 \\ 1 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[R4-R1\to R4]{R1+R2\to R2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & -1 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow[\substack{\frac{1}{2}R2+R3\to R3 \\ \frac{1}{2}R2+R4\to R4}]{\frac{1}{2}R2\to R2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1-R2\to R1} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only the first two columns have pivots $\Rightarrow \{x_1, x_2\}$ is a maximal set of l.i. vectors.

From column 3 of the final matrix: $x_3 = 2x_1 - x_2$

" " 4 " " " " : $x_4 = -x_1 + 2x_2$

2. Consider the plane with parametric representation

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}.$$

Compute the values for the parameters λ and μ which correspond to the point in the plane closest to the given point

$$\boldsymbol{p} = \cancel{\begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}} \begin{pmatrix} -7 \\ -4 \\ 0 \end{pmatrix}.$$

(10)

We need to find the least-square solution of

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & -3 \\ 1 & 1 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} \lambda \\ \mu \end{pmatrix}}_{=:x} = \begin{pmatrix} -7 \\ -4 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}}_{=:b}$$

$$A^T A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 11 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 24 \end{pmatrix}$$

Need to solve $A^T A x = A^T b$, with augmented matrix

$$\left(\begin{array}{cc|c} 2 & -2 & -6 \\ -2 & 11 & 24 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 9 & 18 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 1 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right) \Rightarrow \lambda = -1, \mu = 2$$

3. In how many ways can 5 people be made to stand in a straight line? In a circle? (5)

In a straight line: number of permutations of a 5-element set, ie.

$$5! = 120$$

In a circle: Choice of first person does not matter, so

$$\frac{5!}{5} = 4! = 24$$

4. For a three-child family, let the events E and F be as follows.

   E: The family has at least one boy

   F: The family has children of both sexes

   Find the following.

   (a) $P(E)$
   (b) $P(F)$
   (c) $P(E \cap F)$
   (d) Are E and F independent?

(5)

There are $2^3 = 8$ possibilities altogether.

(a) $P(E) = 1 - P(\text{family has only girls}) = 1 - \frac{1}{8} = \frac{7}{8}$

(b) $P(F) = 1 - P(\text{family has only girls}) - P(\text{family has only boys})$

$= 1 - \frac{1}{8} - \frac{1}{8} = \frac{6}{8}$

(c) $P(E \cap F) = P(F) = \frac{6}{8}$

(d) no, as $P(E \cap F) \neq P(E)\,P(F)$

5. If a basketball player makes 3 out of every 4 free throws, what is the probability that she will make 2 out of 3 free throws in a game? (5)

This is a binomial probability question:

$$p = \frac{3}{4}$$

$$b\left(3, 2 ; \frac{4}{3}\right) = C^3_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^1 = \frac{3!}{2!\,1!} \quad \frac{3^2}{4^3} = \frac{27}{64}$$

6. An oil drilling company has determined that it costs \$10 000 to sink a test well. If oil is hit, the revenue for the company will be \$500 000. If natural gas is found, the revenue will be \$150 000. If the probability of hitting oil is 3% and of hitting gas is 6%, how much should they be willing to pay for the licence to drill? (5)

Expected revenue, in units of \$1000 :

$$\frac{3}{100} \cdot 500 + \frac{6}{100} \cdot 150 = 15 + 9 = 24$$

$\Rightarrow$ expected profit without cost of license: $24 - 10 = 14$

$\Rightarrow$ if the cost of the license remains below 14 k\$, the total expected profit remains positive.

7. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair? (10)

We introduce the following events:

H : coin shows heads

$T = \overline{H}$: coin shows tails

F : coin is fair

$$P(F \mid H) = \frac{P(H \mid F)\ P(F)}{P(H)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$$

8. I have ~~4~~ 2 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

I live in a place where it rains half of the time.

(a) Write out the transition matrix for a Markov chain describing this situation.
Hint: As states, take the number of umbrellas in the place where I am currently at (home or office).

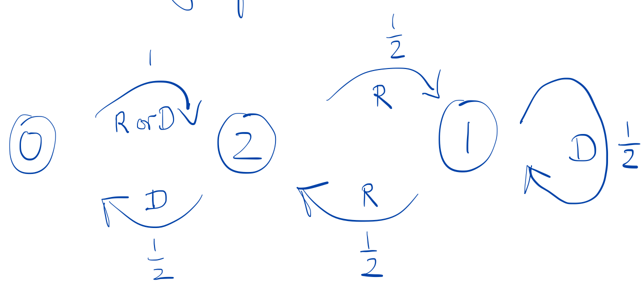
(b) Show that this Markov chain is regular.

(c) Find the stationary distribution.

(d) What is the probability that I get wet on any given walk between home and office?

(5+5+5+5)

(a) Following the hint, we have states 0, 1, 2 with the following transition graph:



R: it rains

D: it is dry

corresponding transition matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

(b) $$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

P^3 has clearly all entries > 0

$\Rightarrow$ chain is regular.

(c) need to solve $P^T x = x$, a homogeneous linear system with matrix

$$\begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -1 \end{pmatrix} \xrightarrow[R1+R3 \to R3]{\substack{-R1 \to R1 \\ -2R2 \to R2}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{R3 - \frac{1}{2}R2 \to R3} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ normalized stationary distribution is $\frac{1}{\frac{1}{2}+1+1} \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$

(d) With probability $\frac{1}{5}$ I don't have an umbrella at hand, so the prob. that I get wet is

$$\frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}.$$

9. An urn contains three black balls and three white balls. A ball is chosen at random without bias. If it is black, it is removed. If it is white, it is replaced. This process is repeated until all black balls are removed.

   (a) Show that this process is an absorbing Markov chain by writing out its transition matrix in canonical form.

   (b) What is the expected number of steps before the process stops?

(5+5)

(a) $$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 2 & 0 & \frac{2}{5} & \frac{3}{5} & 0 \\ 3 & 0 & 0 & \frac{3}{6} & \frac{3}{6} \end{array}$$

$=: R$ (column $\frac{1}{4}, 0, 0$) $\quad =: Q$ (block $\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 \\ 0 & \frac{3}{6} & \frac{3}{6} \end{pmatrix}$)

(b) We need to compute the fundamental matrix $(I-Q)^{-1}$:

$$\left(\underbrace{\begin{matrix} \frac{1}{4} & 0 & 0 \\ -\frac{2}{5} & \frac{2}{5} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{matrix}}_{I-Q} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) \longrightarrow \left(\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{matrix} \;\middle|\; \begin{matrix} 4 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 2 \end{matrix}\right)$$

$$\longrightarrow \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \underbrace{\begin{matrix} 4 & 0 & 0 \\ 4 & \frac{5}{2} & 0 \\ 4 & \frac{5}{2} & 2 \end{matrix}}_{(I-Q)^{-1}}\right)$$

Row sums are the expected exit times from the respective states, here from state 3:

$$4 + \frac{5}{2} + 2 = \frac{17}{2}$$

10. Find the optimal strategies and the value of the game for the two-player zero-sum games represented by the following pay-off matrices:

(a) $G = \begin{pmatrix} -25^{*} & 10 \\ \boxed{10}^{*} & \boxed{50} \end{pmatrix}$

$*$: row minimum

$\square$: column maximum

(b) $G = \begin{pmatrix} \boxed{1} & -2^{*} \\ -3^{*} & \boxed{4} \end{pmatrix}$

(5+5)

(a) The game is strictly determined:

row player plays R2, column player plays C1,

the value of the game is 10.

(b) No saddle point, we must use a mixed strategy:

Expected payoff vector if row player plays $(p, 1-p)$:

$$(p, 1-p) \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} = \big(p - 3(1-p),\ -2p + 4(1-p)\big)$$

$$= (4p - 3,\ -6p + 4)$$

Row player's payoff is independent of choice of column player if

$$4p - 3 = -6p + 4$$

$$\Rightarrow 10p = 7$$

$$\Rightarrow p = \frac{7}{10}$$

$\Rightarrow$ The value of the game is : $4 \cdot \frac{7}{10} - 3 = \frac{28}{10} - 3 = -\frac{1}{5}$

expected payoff vector if column player plays $\begin{pmatrix} q \\ 1-q \end{pmatrix}$:

$$\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} q - 2(1-q) \\ -3q + 4(1-q) \end{pmatrix}$$

$$= \begin{pmatrix} 3q - 2 \\ -7q + 4 \end{pmatrix}$$

Column player's payoff is independent of row player's choice if

$$3q - 2 = -7q + 4$$

$$\Rightarrow \quad 10q = 6$$

$$\Rightarrow \quad q = \frac{6}{10} = \frac{3}{5}$$

(Check expected payoff: $3 \cdot \frac{3}{5} - 2 = -\frac{1}{5}$.)