Finite Mathematics

Makeup Final Exam

August 25, 2020, 17:00–19:00

1. Are the following vectors linearly independent? If not, determine a maximal set of linearly independent vectors, and express the other vectors as linear combinations of vectors from this set.

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}_4 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix}.$$

(10)

2. Consider the plane with parametric representation

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}.$$

Compute the values for the parameters λ and μ which correspond to the point in the plane closest to the given point

$$\boldsymbol{p} = \begin{pmatrix} -7 \\ -4 \\ 0 \end{pmatrix}.$$

(10)

3. In how many ways can 5 people be made to stand in a straight line? In a circle? (5)

4. For a three-child family, let the events E and F be as follows.

   E: The family has at least one boy

   F: The family has children of both sexes

   Find the following.

   (a) $P(E)$

(b) $P(F)$

(c) $P(E \cap F)$

(d) Are E and F independent?

(5)

5. If a basketball player makes 3 out of every 4 free throws, what is the probability that she will make 2 out of 3 free throws in a game? (5)

6. An oil drilling company has determined that it costs \$10 000 to sink a test well. If oil is hit, the revenue for the company will be \$500 000. If natural gas is found, the revenue will be \$150 000. If the probability of hitting oil is 3% and of hitting gas is 6%, how much should they be willing to pay for the licence to drill? (5)

7. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair? (10)

8. I have 2 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

   I live in a place where it rains half of the time.

   (a) Write out the transition matrix for a Markov chain describing this situation.

   Hint: As states, take the number of umbrellas in the place where I am currently at (home or office).

   (b) Show that this Markov chain is regular.

   (c) Find the stationary distribution.

   (d) What is the probability that I get wet on any given walk between home and office?

(5+5+5+5)

9. An urn contains three black balls and three white balls. A ball is chosen at random without bias. If it is black, it is removed. If it is white, it is replaced. This process is repeated until all black balls are removed.

   (a) Show that this process is an absorbing Markov chain by writing out its transition matrix in canonical form.

   (b) What is the expected number of steps before the process stops?

(5+5)

10. Find the optimal strategies and the value of the game for the two-player zero-sum games represented by the following pay-off matrices:

(a) $G = \begin{pmatrix} -25 & 10 \\ 10 & 50 \end{pmatrix}$

(b) $G = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$

(5+5)