Finite Mathematics

Final Exam

May 27, 2020, 16:00–18:00

1. Are the following vectors linearly independent? If not, determine a maximal set of linearly independent vectors, and express the other vectors as linear combinations of vectors from this set.

$$\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 2 \\ -10 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}.$$

(10)

2. Find the best fit of the equation of a straight line, $y = mx + b$, in the least-square sense to the uncertain measurements $y_1 = 1$, $y_2 = 3$, $y_3 = 3$ that correspond to the given, certain values $x_1 = 0$, $x_2 = 1$, $x_3 = 2$. (10)

3. Five distinct points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn? (5)

4. Sarah is traveling home on Deutsche Bahn. Official statistics say that at Hannover, there is a 10% chance that the scheduled connection will be missed, at Berlin there is a 20% chance the scheduled connection will be missed. From past experience, she knows that she won't arrive on the scheduled train home in one out of four trips.

   Is the probability of missing the connection in Berlin independent of the probability of missing the connection in Hannover? If not, what is the probability of missing the connection in Berlin given that she made her connection in Hannover on time? (5)

5. An urn contains two red balls and three green balls. A ball is drawn and replaced. In five such trials, what is the probability that exactly three will come out red? (5)

6. A travel insurance policy provides cancellation cover and emergency return for trips abroad. Past data suggests that 6% of trips are canceled and eligible for a reclaim of the ticket price. In 1% of trips, the traveler is eligible for an emergency return at an average cost of three times the original ticket price. What percentage of the ticket price must be charged for insurance cover, neglecting administrative expenses, so that the insurance company breaks even? (5)

7. A student is taking a multiple choice exam with four given answers per question. On half of the questions, one given answer is so implausible that it can be eliminated without having studied at all, the other answers are equally plausible unless the student knows the correct answer. The student has studied just enough to know the correct answer to half of the questions.

   If the student gives a correct answer, what is the probability that they knew the answer?

   (10)

8. The weather on a given day is classified as dry or rainy. It is observed that "one out of two rainy days is followed by a dry day, and one of out four dry days is followed by a rainy day".

   (a) Find a Markov chain that is consistent with this observation.

   (b) Assuming this model, what is the probability that, given that a day is rainy, the next three days will also be rainy?

   (c) Assuming this model, find the long-term proportion of rainy days.

   (d) Is a Markov chain the only possible model for the given observation? Explain!

   (5+5+5+5)

9. An urn contains three red balls and three green balls. A ball is chosen at random without bias. If it is red, it is removed. If it is green, it is replaced. This process is repeated until all red balls are removed.

   (a) Show that this process is an absorbing Markov chain by writing out its transition matrix in canonical form.

   (b) What is the expected number of steps before the process stops?

   (5+5)

10. Find the optimal strategies and the value of the game for the two-player zero-sum games represented by the following pay-off matrices:

   (a) $G = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$

   (b) $G = \begin{pmatrix} 2 & -2 & 1 & -1 \\ 1 & 2 & 0 & 0 \\ 4 & 2 & -2 & -4 \\ 2 & 1 & -1 & -2 \end{pmatrix}$

   (5+5)