Final Exam solutions

1. v_1 and v_2 are clearly co-linear, with $v_2 = -2v_1$. So let's consider only v_1, v_3 and v_4

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & 2 & 3 \\ 5 & 1 & -4 \end{pmatrix} \xrightarrow[5R1+R3 \to R3]{\substack{-R1 \to R1 \\ -R1+R2 \to R2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & -4 & -4 \end{pmatrix} \xrightarrow[\frac{1}{3}R2 \to R2]{R1-\frac{1}{3}R2 \to R1} \begin{pmatrix} 1 & 0 & \boxed{-1} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So these vectors are still linearly dependent, with

$$\boxed{-1} \cdot v_1 + 1 \cdot v_3 = v_4$$

2. Need to solve

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} m \\ b \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}}_{y}$$

in the least-square sense, i.e., we solve

$$A^T A x = A^T y$$

where $A^T A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$

$$A^T y = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cc|c} 5 & 3 & 9 \\ 3 & 3 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 0 & 2 \\ 1 & 1 & \frac{7}{3} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \frac{4}{3} \end{array}\right)$$

$\Rightarrow \quad y = mx + b$ with $m = 1, \; b = \frac{4}{3}$

3. We need to choose 3 vertices from among the 5 points, so there are $C_3^5 = \frac{5!}{2!\,3!} = \frac{4 \cdot 5}{2!} = 10$ possibilities.

4. Let H be the event of being able to connect in Hannover,
B " " " " " " " " " Berlin,

so $P(H) = \frac{9}{10}$, $P(B) = \frac{8}{10}$

On the other hand

$$P(\text{return on scheduled train home}) = P(B|H)\; P(H)$$

or

$$\frac{3}{4} = P(B|H)\; \frac{9}{10}$$

$$\Rightarrow \quad P(B|H) = \frac{5}{6} \neq \frac{8}{10} = P(B)$$

$\Rightarrow$ H and B are not independent, and $P(\overline{B}\,|H) = 1 - \frac{5}{6} = \frac{1}{6}$.

5. This is a Bernoulli trial, and

$$P(\text{three balls are red}) = P\left(5, 3; \tfrac{2}{5}\right) = C_3^5 \left(\tfrac{2}{5}\right)^3 \left(\tfrac{3}{5}\right)^2$$

$$= \frac{5!}{3!\,2!} \cdot \frac{8 \cdot 9}{5^5} = 2 \cdot \frac{8 \cdot 9}{5^4} = \frac{144}{625}$$

6. Let p be the price of some ticket.
Expected loss on that ticket: $L = \frac{6}{100} \cdot p + \frac{1}{100} \cdot 3p$

$$= \frac{9}{100} \cdot P$$

The insurance company should charge 9% of the ticket price to cover expected losses.

7. Let K be the event that the student knows the answer,
C " " " " " answers correctly.

$$P(C) = \underbrace{P(C|K)}_{=1}\underbrace{P(K)}_{=\frac{1}{2}} + \underbrace{P(C|\bar{K})}_{=\frac{\frac{1}{4}+\frac{1}{3}}{2}}\underbrace{P(\bar{K})}_{=\frac{1}{2}}$$

$$= \frac{1}{2} + \frac{1}{16} + \frac{1}{12} = \frac{24+3+4}{48} = \frac{31}{48}$$

$$P(K|C) = \frac{P(C|K)\,P(K)}{P(C)} = \frac{\frac{1}{2}}{\frac{31}{48}} = \frac{24}{31} \approx 0.774$$

8. (a) Transition matrix

$$P = \begin{array}{c} \\ R \\ D \end{array} \begin{array}{c} \begin{array}{cc} R & D \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \end{array}$$

(b) $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(c) Stationary distribution satisfies

$$x^T = x^T P \qquad \text{or} \qquad \underbrace{(P^T - I)}_{\begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \end{pmatrix}} x = 0$$

$$= \begin{pmatrix} \cdots & \cdots \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

The matrix has rank 1 by inspection, and homogeneous solutions are multiples of $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

=> 2 out of 3 days are dry
1 out of 3 days are rainy

(d) A Markov chain is obviously only one of many models that can produce the stated observations. It is the model where everything NOT observed is assumed independently random and unbiased.
But the same observations can also be produced by a sequence is periodic, e.g. RRDDDD... repeated forever, hence is entirely rigid, or a more complex model that tries to include some of underlying physics of the problem beyond what is observed.

9. (a) The states are the number of red balls left, with 0 being the absorbing state. Transition matrix:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array} \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 0 & \frac{3}{6} & \frac{3}{6} \end{pmatrix}$$

$=: R$ (column 0, rows 1–3) $\qquad =: Q$ (columns 1–3, rows 1–3)

(b) The fundamental matrix is $(I - Q)^{-1}$. We use Gaussian

(b) The fundamental matrix is $(I-Q)^{-1}$. We use Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} \frac{1}{4} & 0 & 0 & 1 & 0 & 0 \\ -\frac{2}{5} & \frac{2}{5} & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & 0 \\ -1 & 1 & 0 & 0 & \frac{5}{2} & 0 \\ 0 & -1 & 1 & 0 & 0 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 4 & \frac{5}{2} & 0 \\ 0 & 0 & 1 & 4 & \frac{5}{2} & 2 \end{array}\right) \Rightarrow (I-Q)^{-1} = \begin{pmatrix} 4 & 0 & 0 \\ 4 & \frac{5}{2} & 0 \\ 4 & \frac{5}{2} & 2 \end{pmatrix}$$

The row sums are the expected exit times from each of the non-absorbing states, i.e. $\begin{pmatrix} 4 \\ \frac{13}{2} \\ \frac{17}{2} \end{pmatrix}$

So when starting with 3 red balls, we will take, on average $\frac{17}{2} = 8.5$ steps to exit.

10. (a) $$G = \begin{pmatrix} \boxed{1} & -2^{*} \\ -1^{*} & \boxed{2} \end{pmatrix}$$

$*$ row minimum

$\square$ column maximum

$\Rightarrow$ no saddle point, the optimal strategy is mixed.

Expected pay-off for the row player for each choice of column player:

$$(p,\ 1-p)\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} = \big(p-(1-p),\ -2p+2(1-p)\big)$$
$$= (2p-1,\ 2-4p)$$

For optimal strategy, both must co-incide, i.e.

$$2p-1 = 2-4p \Rightarrow 6p = 3 \Rightarrow p = \frac{1}{2}$$

with expected pay-off $2p-1 = 0 \Rightarrow$ the game is fair.

Expected pay-off for the column player for each choice of the row player:

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} q - 2(1-q) \\ -q + 2(1-q) \end{pmatrix} = \begin{pmatrix} 3q-2 \\ 2-3q \end{pmatrix}$$

Optimal strategy:

$$3q-2 = 2-3q \Rightarrow 6q = 4 \Rightarrow q = \frac{2}{3}$$

(Check expected payoff: $3q-2 = 3 \cdot \frac{2}{3} - 2 = 0$ ✓)

(b) Eliminate by dominance:

$$G = \begin{pmatrix} 2 & -2^* & \boxed{1} & -1 \\ 1 & \boxed{2} & 0^* & \boxed{0}^* \\ \boxed{4} & \boxed{2} & -2 & -4^* \\ 2 & 1 & -1 & -2^* \end{pmatrix}$$

The matrix has a saddle point at R2, C4, so that the game is strictly determined. Note that is possible, but not necessary, to eliminate C1, C3, R1, R3 by dominance first.