Advanced Calculus and Methods of Mathematical Physics

Homework 9

Due via Gradescope, Friday, May 9, before 20:00

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions can be found in the appendix of Kantorovitz' book.

1. Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$.

   (a) The *"BAC–CAB-identity"*

   $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})\,.$$

   (b) The *Jacobi identity* in three dimensions

   $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0\,.$$

   (c) The *Cauchy–Binet formula* in three dimensions

   $$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c})\,.$$

   All three identities are important and worth memorizing. Note that (c) directly implies the identity

   $$\|\boldsymbol{a} \times \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$$

   we used in class.

2. *(Kantorovitz, Exercises 4.5.7, Problem 6.) A smooth curve is given in the yz-plane by the parameterization

   $$\gamma(t) = (0, y(t), z(t))\,, \quad t \in [a, b]\,.$$

   The surface M is obtained by revolving γ about the z-axis.

   (a) Show that M has surface area

   $$\sigma(M) = 2\pi \int_\gamma |y|\,\mathrm{d}s = 2\pi \int_a^b |y(t)|\,\|\gamma'(t)\|\,\mathrm{d}t\,.$$

(b) Take γ to be the circle centered at $(0, R, 0)$ with radius $r \in (0, R)$ in the yz-plane, so that M is a torus. Find the area of the torus M.

3. Prove *Pappus' theorem*: If M is the surface obtained by revolving a plane curve γ of length $L = \Lambda(\gamma)$ about an axis in the plane of the curve at distance h from the centroid of the curve, then

$$\sigma(M) = 2\pi L h\,.$$

Hint: Use the formula from Question 2(a).

4. Show, by explicit computation, that the surface area of a smooth regular surface M with parameterization $f \in C^1(U \to \mathbb{R}^3)$,

$$\sigma(M) = \int_U \|n\|\,\mathrm{d}S = \int_U \left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\| \mathrm{d}u$$

is independent of the parameterization. I.e., if $g \in C^1(V, \mathbb{R}^3)$ is another smooth regular parameterization with $g = f \circ \phi$ for some $phi \in C^1(V, U)$, then

$$\sigma(M) = \int_V \left\| \frac{\partial g}{\partial v_1} \times \frac{\partial g}{\partial v_2} \right\| \mathrm{d}v\,.$$

Hint: Use chain rule, change-of-variable formula, and the properties of the cross-product.

5. *(Kantorovitz, Exercises 4.5.7, Problem 4.) Let $F = (xy, 0, -z^2)$, $D = [0,1]^3$, and $M = \partial D$ oriented such that the normal vector points outwards. Calculate the flux

$$\Phi = \int_M F \cdot n\,\mathrm{d}\sigma$$

(a) by applying the divergence theorem,

(b) directly.

6. *(Kantorovitz, Exercises 4.5.7, Problem 7.) Let γ be the closed curve parameterized by

$$\gamma(t) = (\cos t, \sin t, \cos 2t)\,, \quad t \in [0, 2\pi]\,.$$

Let M be the portion of the hyperbolic paraboloid S defined by the equation

$$z = x^2 - y^2$$

with boundary γ (note that γ lies on S!). Calculate the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

for the vector field $F = (x^2 + z^2, y, z)$

(a) by applying Stokes' theorem,

(b) directly.

7. Use Stokes' theorem to compute the line integral

$$\int_\gamma x^2\, y^3\, \mathrm{d}x + \mathrm{d}y + z\, \mathrm{d}z$$

where γ is the circle $x^2 + y^2 = a^2$ in the xy-plane.

8. Let $D \subset \mathbb{R}^k$ be the simplex with k of its faces faces on the coordinate hyperplanes, namely

$$S_j \subset \{x \in \mathbb{R}^k \colon x_j = 0\}\,, \quad j = 1, \ldots, k\,,$$

and the final face S in the interior of the first orthant, i.e.,

$$S \subset \{x \in \mathbb{R}^k \colon x_i \geq 0, i = 1, \ldots, k\}\,.$$

Show that the outward normal vector n on S has coordinates

$$n_j = \frac{\sigma(S_j)}{\sigma(S)}\,.$$

Hint: Use the divergence theorem for smartly chosen vector fields.