Advanced Calculus and Methods of Mathematical Physics

Homework 5

Due in class Tuesday, March 10, 2020

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions will be made available.

Inverse Function Theorem. *For $E \subset \mathbb{R}^n$ open and $f \in C^1(E, \mathbb{R}^n)$, suppose that $Df(a)$ is invertible for some $a \in E$.*[1] *Then*

(i) There exist open sets $U, V \subset \mathbb{R}^n$ with $a \in U$ such that $f\colon U \to V$ is bijective,

(ii) $f^{-1} \in C^1(V, U)$ and
$$Df^{-1}(y) = Df(x)^{-1} \tag{*}$$
with $y = f(x)$.

Implicit Function Theorem. *Let $X \subset \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be open and $f \in C^1(X \times Y, \mathbb{R}^n)$. Suppose there exist $a \in X$ and $b \in Y$ such that $f(a, b) = 0$ and $D_x f(a, b)$ is invertible. Then there exist an open neighborhood A of a, an open neighborhood B of b, and a function $g \in C^1(B, A)$ such that*
$$f(g(y), y) = 0$$
for every $y \in B$, and
$$Dg(y) = -D_x f(g(y), y)^{-1}\, D_y f(g(y), y)\,.$$

1. (From Rudin, Exercise 9.17.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f(x) = \begin{pmatrix} \mathrm{e}^{x_1} \, \cos x_2 \\ \mathrm{e}^{x_1} \, \sin x_2 \end{pmatrix}\,.$$
   (a) What is the range of f?
   (b) Show that the *Jacobian determinant*, $\det Df(x)$, is non-zero for every $x \in \mathbb{R}^2$. Thus every point in $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.

[1] Here, DF denotes the matrix representation of the derivative $\mathrm{d}f$. Usually, we do not make this distinction explicit, but in this context, the focus is on the block matrix structure so that, following convention, the notation DF, $D_x f$, and $D_y f$ is preferred.

(c) Put $a = (0, \pi/3)$ and $b = f(a)$. Find an explicit formula for f^{-1}, compute $Df(a)$ and $Df^{-1}(b)$, and verify formula (*) above.

(d) What are the images under f of lines parallel to the coordinate axes?

2. *(From the *inverse* to the *implicit function theorem*.) Let $X \subset \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be open and $f \in C^1(X \times Y, \mathbb{R}^n)$. Define $F \colon X \times Y \to \mathbb{R}^n \times \mathbb{R}^m$ as

$$F(x, y) = \begin{pmatrix} f(x, y) \\ y \end{pmatrix} .$$

(a) Show that DF has the block-matrix structure

$$DF = \begin{pmatrix} D_x f & D_y f \\ 0 & I_m \end{pmatrix} ,$$

where I_m is the $m \times m$-identity matrix and $D_x f$ and $D_y f$ denote the generalized partial derivatives

$$D_x f = \begin{pmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_n} f_1 \\ \vdots & & \vdots \\ \partial_{x_1} f_n & \dots & \partial_{x_n} f_n \end{pmatrix} \quad \text{and} \quad D_y f = \begin{pmatrix} \partial_{y_1} f_1 & \dots & \partial_{y_m} f_1 \\ \vdots & & \vdots \\ \partial_{y_1} f_n & \dots & \partial_{y_m} f_n \end{pmatrix} .$$

(b) Conclude that DF is invertible if and only if $D_x f$ is invertible.

(c) Suppose that there exist $a \in X$ and $b \in Y$ such that $f(a, b) = 0$ and $D_x f(a, b)$ is invertible. Argue that there exists an open neighborhood V of $(0, b)$ and an open neighborhood U of (a, b) such that $F \colon U \to V$ is bijective with $F^{-1} \in C^1(V, U)$.

(d) Conclude that there exist an open neighborhood A of a, an open neighborhood B of b, and a function $g \colon B \to A$ such that

$$f(g(y), y) = 0$$

for every $y \in B$.

(e) Finally, argue that $g \in C^1(B, A)$ with

$$Dg(y) = -D_x f(g(y), y)^{-1} \, D_y f(g(y), y) .$$

3. *(Kantorovitz, p. 106, Exercise 1.) Show that the equation

$$x^5 + y^5 + z^5 = 2 + xyz$$

determines in a neighborhood of the point $(1, 1, 1)$ a unique function $z = z(x, y)$ of class C^1, and calculate its partial derivatives with respect to x and y at the point $(1, 1)$.

4. Consider the equation

$$\sqrt{x^2 + y^2 + 2\,z^2} = \cos z$$

near $(0, 1, 0)$. Can you solve for x in terms of y and z? For z in terms of x and y?

5. Show that if r is a simple root of the polynomial

$$p(x) = a_0 + a_1\,x + \cdots + a_n\,x^n\,,$$

then r is a C^1 function of the coefficients $a_0, \ldots, a_n$.

6. *Argue that, in the implicit function theorem, if $f \in C^k(X \times Y, \mathbb{R}^n)$ for $k \geq 1$, then $g \in C^k(B, A)$. Conclude that in Problem 5, the dependence is not only C^1, but smooth (i.e., C^∞).

   Hint: Refer to Cramer's rule for the inverse matrix.