Advanced Calculus and Methods of Mathematical Physics

Homework 4

Due in class Tuesday, March 3, 2020

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions will be made available.

1. *(Kantorovitz, p. 78, Exercise 6.) Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be defined by

$$f(x) = \sum_{i=1}^{k} (x_i, x_i^2, \dots, x_i^m)\,.$$

Compute the derivative $\mathrm{d}f(x)$.

2. *(Kantorovitz, p. 78, Exercise 8.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable on $\mathbb{R}^2 \setminus \{0\}$. Let

$$h(r,\theta) = (r\,\cos\theta, r\,\sin\theta)$$

denote the change from polar to Cartesian coordinates and set $g = f \circ h$. Prove that, for $r > 0$,

$$\|(\nabla f) \circ h\|^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \left(\frac{1}{r}\,\frac{\partial g}{\partial \theta}\right)^2.$$

3. *(Kantorovitz, p. 79, Exercise 9.) Prove or disprove the differentiability of the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$.

   (a) $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{\|(x,y)\|} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$

   (b) $f(x,y) = (xy)^{2/3}$

4. Show that the function

$$u(x) = \ln\|x\|$$

solves the Laplace equation

$$\Delta u = 0$$

on $\mathbb{R}^2 \setminus \{0\}$.

5. Let V and W be normed vector spaces and $f\colon V \to W$ be continuously differentiable and homogeneous of degree $\alpha > 0$, i.e., for every $x \in V$ and $t > 0$,

$$f(tx) = t^\alpha f(x)\,.$$

Show that

$$\mathrm{d}f(x)x = \alpha\, f(x)\,.$$

6. Let V be a normed vector space, $E \subset V$ open, and $f\colon E \to \mathbb{R}$ differentiable on E. Suppose that f has a local maximum at some point $x \in E$. Show that, for every $\boldsymbol{v} \in V$,

$$D_{\boldsymbol{v}} f(x) = 0\,.$$

Remark: If $V \in \mathbb{R}^n$, this implies that

$$\nabla f(x) \equiv (\partial_1 f(x), \ldots, \partial_n f(x)) = 0\,.$$

7. Let V and W be normed vector spaces and let $A\colon V \to W$ and $B\colon W \to V$ be bounded linear operators. Furthermore, suppose that

$$\|I - BA\|_{L(V)} < 1\,.$$

(a) Show that $F\colon L(W,V) \to L(W,V)$, defined for every $X \in L(W,V)$ by

$$FX = X + B - BAX\,,$$

is a contraction.

(b) Conclude that F has a fixed point X^*. State an upper bound for the operator norm $\|X^*\|_{L(W,V)}$.