Advanced Calculus and Methods of Mathematical Physics

Homework 3

Due in class Tuesday, February 25, 2020

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions will be made available.

1. *Find a power series representation centered at 2 for
$$\frac{1}{4\,x - x^2 - 3}\,.$$
What is the radius of convergence?

2. *Let X, Y be metric spaces, $E \subset X$ compact, and $f\colon X \to Y$ continuous. Show that $f(E)$ is a compact subset of Y.

3. Recall the operator norm for maps A between normed vector spaces X and Y, defined by
$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}\,.$$
   (a) Verify that the operator norm is indeed a norm.

   (b) Let X, Y, and Z be normed vector spaces, and let $B\colon X \to Y$ and $A\colon Y \to Z$ be linear maps. Show that
$$\|AB\| \leq \|A\|\,\|B\|\,.$$

(4)

4. *Disconcerting Example 1.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$
   (a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

(b) Show that f is not differentiable at the origin.

(2)

5. *Disconcerting Example 2.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3\,y}{x^6+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

(b) Show that f is not continuous at the origin.

(2)

6. *Disconcerting Example 3.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2\,y}{x^4+y^2}\,\sqrt{x^2+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

(b) Show that f is not differentiable at the origin.

(2)