Advanced Calculus and Methods of Mathematical Physics

Homework 2

Due in class Tuesday, February 18, 2020

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions will be made available.

1. *Show that, for $|x| < 1$,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} .$$

   Hint: Represent $\arctan x$ by an integral.

2. Find a power series expansion centered at 0 for

$$\frac{1}{(1+x^2)^2}$$

   and determine the radius of convergence. (2)

3. Test the following series for uniform convergence on $\mathbb{R}$:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$

   (3)

4. (Rudin, Exercise 7.9.) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

   for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. (2)

5. As in the lecture on February 14, let $\alpha \in R$ and define the function

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

on the interval $(-1, 1)$.

(a) Show that

$$f'(x) = \frac{\alpha}{1+x} f(x) \,.$$

Justify the steps of your argument.

(b) Show that

$$g(x) = \frac{f(x)}{(1+x)^\alpha}$$

equals the constant 1.

(c) Conclude that $f(x) = (1+x)^\alpha$ on $(-1, 1)$.

(3)

6. *Suppose the series of complex numbers

$$\sum_{n=0}^{\infty} c_n$$

is convergent. Show that the power series

$$\sum_{n=0}^{\infty} c_n \, x^n$$

converges uniformly (but not necessarily absolutely!) for $x \in [0, 1]$.

Hints: Consider the "remainder" series

$$r_n = \sum_{k=n}^{\infty} c_n$$

and express the c_n in terms of r_n. Using this relation, estimate

$$\sum_{n=\ell}^{m} c_n \, x^n \,.$$

You will encounter a geometric series as you go along, which provides the uniformity on the interval $x \in [0, 1]$.