Advanced Calculus and Methods of Mathematical Physics

Homework 1

Due in class Tuesday, February 11, 2020

Note: Assignments marked (*) are *not* for bonus credit. They will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions will be made available.

1. Recall the mean value theorem of integral calculus: Suppose $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable and non-negative, and $f\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $\xi \in [a,b]$ such that
$$f(\xi) \int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)\,g(x)\,\mathrm{d}x\,.$$
Give an example each to show that the following assumptions cannot be generally dropped:
    (a) g is non-negative,
    (b) f is continuous.

2. *Recall Taylor's theorem in the following form. Suppose $f \in C^{n+1}(I)$ for some open interval I. Then for all $a, x \in I$,
$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}}{j!}\,(x-a)^j + R_n(x)$$
where
$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n\, f^{(n+1)}(t)\,\mathrm{d}t\,.$$
    (a) Turn the derivation shown in class into a formal proof by induction.
    (b) Show that the remainder can also be written as
$$R_n(x) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-s)^n\, f^{(n+1)}(a+s(x-a))\,\mathrm{d}s\,.$$

(c) Show that there exists $\xi \in [a, x]$ such that

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} \, f^{(n+1)}(\xi) \, .$$

(This expression is known as the *Lagrange form* of the remainder.)

(3)

3. *Show that

$$f_n(x) = \frac{\sin nx}{n}$$

is differentiable for every $n \in \mathbb{N}$. Then show that this sequence tends to 0 pointwise, but the pointwise limit of the f_n' does not exist for some $x \in \mathbb{R}$.

4. Find the Maclaurin series (i.e., the Taylor series centered at $a = 0$) of the function

$$f(x) = \mathrm{e}^{-x^2} \, .$$

What it the interval of convergence? Does it converge to the function f and why? (4)

5. Find the limit of the series

$$\sum_{n=1}^{\infty} n^3 \, x^n \, .$$

What is the radius of convergence? (3)

6. *Consider a sequence of continuous functions $f_n \colon [a, b] \to [c, d]$ that converges uniformly to a function $f \colon [a, b] \to \mathbb{R}$. Let ϕ be continuous on $[c, d]$. Show that the sequence $g_n = \phi \circ f_n$ converges uniformly to $g = \phi \circ f$.