Introduction to complex variables

$D \subset \mathbb{C}$ domain

$f : D \to \mathbb{C}$ is differentiable (holomorphic) at $z = z_0$, if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

$$\Rightarrow \boxed{\begin{array}{c} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{array}}$$

Cauchy - Riemann relations

Notation: $f = u + iv \qquad u, v : D \to \mathbb{R}$

$z = x + iy$

Necessary condition:

$$\underbrace{\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}}_{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}} = \underbrace{\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\underline{i} \Delta y}}_{-i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}}$$

Note: C-R relations imply complex-differentiability at $z = z_0$ if the partial derivatives are C^1 near a point z_0.

Complex line integral:

$\gamma \subset D$ curve

$$\int_\gamma f(z)\, dz = \int_\gamma (u + iv)(dx + i dy) = \int_\gamma u\, dx - v\, dy + i \int_\gamma v\, dx + u\, dy$$

$$= \int_\gamma F(\underline{X}) \cdot d\underline{X} + i \int_\gamma G \cdot d\underline{X}$$

$\underline{X} = (x, y)$

$F = \begin{pmatrix} u \\ -v \end{pmatrix}$

$G = \begin{pmatrix} v \\ u \end{pmatrix}$

Suppose that γ is a simple closed curve, $S \subset D$, $\partial S = \gamma$

$$\ldots = \int_S \nabla^\perp \cdot F \, dS + i \int_S \nabla^\perp \cdot G \, dS = 0,$$ see next page

$$\int_\gamma F \cdot dx \underset{\substack{\uparrow \\ \text{Green's}}}{=} \int_S \nabla^\perp \cdot F \, dS$$

$$F = \begin{pmatrix} u \\ -v \end{pmatrix}$$

$$\nabla^\perp \cdot F = \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} u \\ -v \end{pmatrix} = -\frac{\partial u}{\partial y} - \underbrace{\frac{\partial v}{\partial x}}_{\overset{C.R.}{=} -\frac{\partial u}{\partial y}} = 0$$

if f is complex-differentiable

$$\int_\gamma G \cdot dx \underset{\substack{\uparrow \\ \text{Green's}}}{=} \int_S \nabla^\perp \cdot G \, dS$$

$$G = \begin{pmatrix} v \\ u \end{pmatrix}$$

$$\nabla^\perp \cdot G = -\frac{\partial v}{\partial y} + \underbrace{\frac{\partial u}{\partial x}}_{= \frac{\partial v}{\partial y}} = 0$$

We have proved the following:

Cauchy's Theorem: $f : D \to \mathbb{C}$ holomorphic

$\gamma \subset D$ closed curve

$$\Rightarrow \int_\gamma f(z)\, dz = 0$$

"line integrals of holomorphic functions can be identified with line integrals of conservative vector fields"

Cor. Cauchy's integral formula: $f : D \to \mathbb{C}$ holomorphic

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \qquad \text{if } D(z,r) \subset D$$

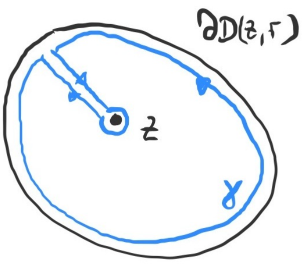
Proof: $$\int_{\partial D(z,r)} \frac{1}{\zeta - z}\, d\zeta = \int_{\partial D(0,r)} \frac{1}{\zeta}\, d\zeta$$ (shift into origin)

$$= \int_0^{2\pi} \frac{i\cancel{r}\cancel{e^{i\theta}}\, d\theta}{\cancel{r}\cancel{e^{i\theta}}}$$

$\zeta = r e^{i\theta}$ (complex polar coordinates)

$d\zeta = i r e^{i\theta}\, d\theta$

$$= 2\pi i$$



By Cauchy: $$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0$$

$$\parallel$$

$$\int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial B(z,\delta)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$\xrightarrow[\delta \to 0]{} f(z) \cdot 2\pi i$$

□

Note: The previous is true, if γ is any closed simple curve encircling the point γ.

Cor.: under the same assumptions:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Cor.: f holomorphic on D $\Rightarrow$ $f \in C^\infty$

Cor.: f holomorphic on D $\Rightarrow$ f is analytic

f has a convergent Taylor series

Proof: $$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z}$$

$$= \frac{1}{\zeta - z_0} \underbrace{\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}}_{\sum_{k=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k}$$

with radius of convergence

$$\left|\frac{z-z_0}{\zeta - z_0}\right| < 1$$

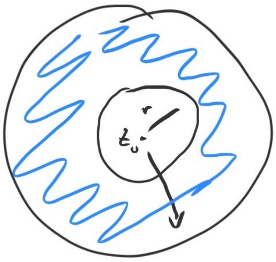
(uniformly on any disk with rad. < 1)

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{\cancel{2\pi i}} \sum_{k=0}^{\infty} (z-z_0)^k \underbrace{\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta}_{= \frac{\cancel{2\pi i}}{k!} f^{(k)}(z_0)} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k$$

□

Laurent series:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$



Typical domain of convergence is an annulus

Interesting case: inner radius of convergence is 0.

"f has an isolated singularity"

Then:

$$\int_{\partial D(z_0,r)} f(z)\, dz = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{\partial D(z_0,r)} (z-z_0)^k dz}$$

$$z = z_0 + re^{i\theta}$$

$$dz = ire^{i\theta} d\theta$$

$$= \int_0^{2\pi} (re^{i\theta})^k \, ire^{i\theta} d\theta$$

$$= ir^{k+1} \overset{= 2\pi \delta_{k,-1}}{=} \int_0^{2\pi} e^{i(k+1)\theta} d\theta = 2\pi i \overset{\overset{\text{Res}(f, z_0)}{\|}}{a_{-1}}$$

We have thus proved the following:

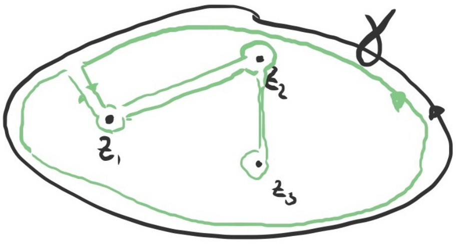
Residual Theorem: $D \subset \mathbb{C}$ simply connected domain

f is holomorphic (analytic) except at a finite number of isolated points $z_1, \dots, z_m$

γ simple closed curve enclosing all z_k

$$\Rightarrow \boxed{\int_\gamma f(z)\, dz = 2\pi i \sum_{n=1}^{m} \operatorname{Res}(f, z_n)}$$

Proof by picture:



Complex Variables:

See Chapter 24 of Riley/Hobson/Bence

Fourier:

Will send link to Lecture Notes