Fourier series

2π-periodic : $f : [0,2\pi] \to \mathbb{C}$ $\quad f(0) = f(2\pi)$ $\quad$ R-integrable on $[0,2\pi]$

$[0,2\pi)$ $\quad f(x+2\pi) = f(x)$

Goal: Decomposition into "pure frequencies"

$$f(x) = \sum_{k\in\mathbb{Z}} \gamma_k e^{ikx} \qquad (*) \qquad e^{ikx} = \cos kx + i \sin kx$$

"basis function" $e_k(x) = e^{ikx}$

expansion coefficients or Fourier coefficients.

Introduce inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\, g(x)\, dx \qquad \|f\|^2 = \langle f, f\rangle$$

$$\langle e_j, e_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijx} e^{ikx}\, dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-j)x}\, dx = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases}$$

Take inner product of $(*)$ with e_j

$$\langle e_j, f\rangle = \langle e_j, \sum_{k\in\mathbb{Z}} \gamma_k e_k\rangle \overset{\text{"}\,\text{"}}{=} \sum_{k\in\mathbb{Z}} \gamma_k \underbrace{\langle e_j, e_k\rangle}_{\delta_{jk}} = \gamma_j$$

This motivate the definition of the Fourier transform

$$f_k = \langle e_k, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\, dx$$

and the Fourier series or inverse transform

$$f(x) = \sum_{k\in\mathbb{Z}} f_k e^{ikx}$$

Note: IF Fourier series is uniformly convergent, then we can swap limit and integral in "=", and everything is rigorous.

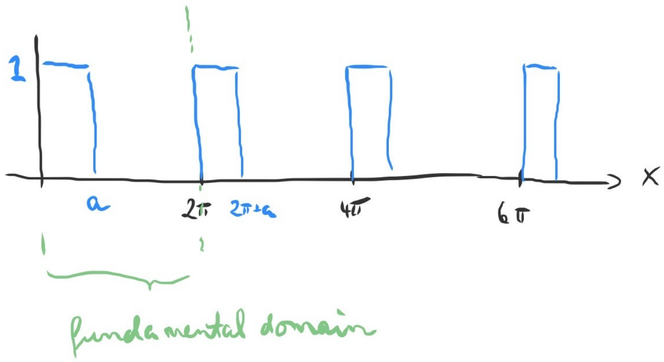
Examples: ① $f(x) = 1$

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{e^{i0x}}_{1}\, 1\, dx = 1$$

In general: f_0 is the average of f over $[0,2\pi]$

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\, dx = 0 \quad \text{for } k \neq 0$$

② $f(x) =$



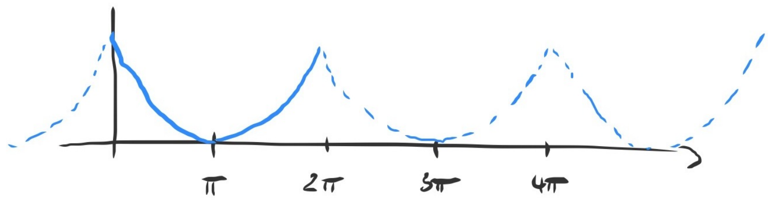
$$f_0 = \frac{a}{2\pi}$$

$$k \neq 0: \quad f_k = \frac{1}{2\pi}\underbrace{\int_0^a e^{-ikx}\, dx}_{} = \frac{i}{2\pi k}\left(e^{-ika} - 1\right)$$

$$= \left.\frac{e^{-ikx}}{-ik}\right|_0^a = \frac{i}{k}\left(e^{-ika} - 1\right)$$

Note: $|f_k| \sim \frac{1}{k}$ so Weierstrass test for uniform convergence fails (in fact, Fourier series does not converge uniformly → old HW problem, difficult)

③ $f(x) = (x-\pi)^2$ on $[0, 2\pi]$



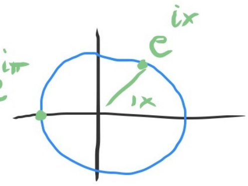
$$f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}(x-\pi)^2\,dx$$

$y = x - \pi \qquad dy = dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-ik(y+\pi)}}_{e^{-iky}\,\underbrace{e^{-ik\pi}}_{=(-1)^k}} y^2\,dy$$

$e^{ik\pi} = e^{i\pi}$

k odd



$$= \frac{(-1)^k}{2\pi}\underbrace{\int_{-\pi}^{\pi} e^{-iky}\,y^2\,dy}_{(*)}$$

$$(*) = \underbrace{\frac{e^{-iky}}{-ik}\,y^2\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{-ik}\int_{-\pi}^{\pi} e^{-iky}\,2y\,dy \qquad k \neq 0$$

$$= \frac{2}{k^2}\,e^{-iky}\,y\Big|_{-\pi}^{\pi} - \frac{2}{k^2}\underbrace{\int_{-\pi}^{\pi} e^{-iky}\,1\,dy}_{=0}$$

$$= \frac{2\pi}{k^2}\left(e^{-ik\pi} - (-e^{ik\pi})\right) = \frac{4\pi}{k^2}(-1)^k \quad \Rightarrow \quad f_k = \frac{(-1)^k}{2\pi}\,\frac{4\pi}{k^2}(-1)^k = \frac{2}{k^2}$$

$$f_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^2\,dy = \ldots = \frac{1}{3}\pi^2$$

$$(x-\pi)^2 = \underbrace{\frac{1}{3}\pi^2}_{f_0} + \underbrace{\sum_{k\neq 0} \frac{2}{k^2} e^{ikx}}$$

$$= \sum_{k=1}^{\infty} \frac{2}{k^2}\left(e^{ikx} + e^{-ikx}\right) = \sum_{k=1}^{\infty} \frac{4}{k^2} \cos kx$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}}$$

Note: this converges uniformly according to Weierstrass

$x=0$: $$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}}$$

Q: Convergence for functions no better than R-integrable (like ②)

Idea: In general, measure convergence by norm coming from our inner product:

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \, dx$$

"mean-square convergence"

"L^2-convergence"

$$\Big\| f - \sum_{|k|\leq n} f_k e_k \Big\|^2 = \|f\|^2 - \sum_{|k|\leq n} \underbrace{\left(\langle f, f_k e_k\rangle + \langle f_k e_k, f\rangle\right)}_{= \bar{f_k}\underbrace{\langle f, e_k\rangle}_{f_k} + f_k \underbrace{\langle e_k, f\rangle}_{\bar{f_k}}} + \sum_{\substack{|k|\leq n \\ |j|\leq n}} \underbrace{\langle f_k e_k, f_j e_j\rangle}_{f_k \bar{f_j} \delta_{jk}}$$

$$= \|f\|^2 - \sum_{|k|\leq n} |f_k|^2$$

For example (2):

$$\|f\| = \frac{a}{2\pi}$$

$$\sum_{k\in\mathbb{Z}} |f_k|^2 = \underbrace{\left(\frac{a}{2\pi}\right)^2}_{|f_0|^2} + \underbrace{\sum_{k\neq 0} \underbrace{\frac{1-e^{-ika}}{2\pi k}}_{\overline{f_k}} \; \underbrace{\frac{1-e^{ika}}{2\pi k}}_{f_k}}$$

$$= \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1-\cos ka}{k^2} \underbrace{= \dots =}_{\text{Example (3)}} \frac{1}{4} - \frac{a^2 - 2a\pi + \pi^2}{4\pi^2}$$

$$= \frac{a}{2\pi}$$

$\Rightarrow$ Fourier series converges mean-square

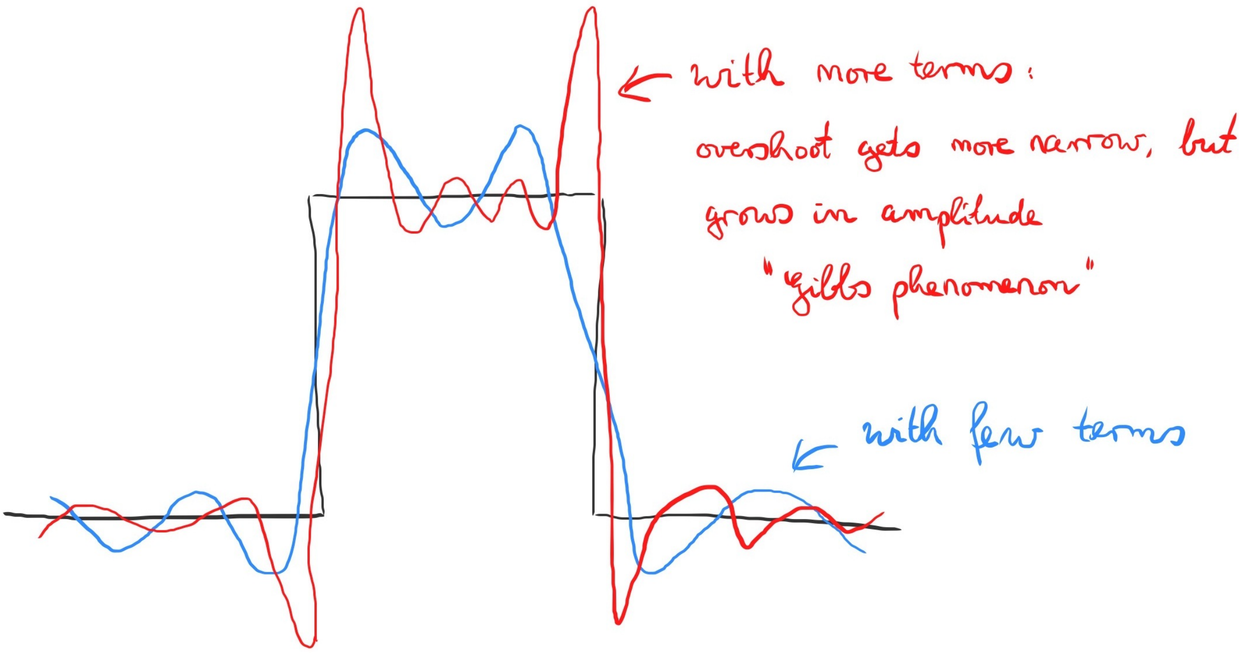
Theorem: $f : [0, 2\pi] \to \mathbb{C}$ periodic, R-integrable

$\Rightarrow$ Fourier series is mean-square convergent and

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \, dx = \sum_{k\in\mathbb{Z}} |f_k|^2 \qquad \text{"Parseval identity"}$$

Proof: Approximate any R-integrable function by a linear combination of square pulses. For those, the computation of Example 2 applies, take limit of an approximating sequence.

Approximation of square pulse by a truncated Fourier Series:



- For discontinuous functions: no uniform convergence, but mean-square convergence.