Green's Theorem:

$$\int_D \underbrace{\nabla^\perp \cdot F}_{=\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}} \, dS = \int_{\partial D} F \cdot dx \qquad \text{(anti-clockwise orientation)}$$

<u>Example:</u> Compute $J = \int_D xy \, dS$

where D is bounded by the ellipse $\gamma : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $x, y \geq 0$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

(a) $$J = \int_0^a \underbrace{\int_0^{b\sqrt{1 - x^2/a^2}} xy \, dy}_{= x \cdot \frac{1}{2} y^2 \Big|_{y=0}^{y = b\sqrt{1 - x^2/a^2}} = x \cdot \frac{1}{2} b^2 \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{2}\left(x - \frac{1}{a^2} x^3\right)} \, dx$$

$$= \frac{b^2}{2} \int_0^a \left(x - \frac{1}{a^2} x^3\right) dx = \frac{b^2}{2}\left(\frac{1}{2} a^2 - \frac{1}{4} \frac{a^4}{a^2}\right) = \frac{a^2 b^2}{8}$$

(b) Using Green's Theorem:

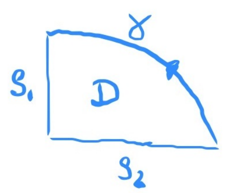
Need vector field F s.t. $\nabla^\perp \cdot F = xy$, e.g. $F = \left(0, \frac{1}{2} x^2 y\right)$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = xy - 0$$

$J = \int_{\partial D} F \cdot dx$ Need to parameterize γ: $\gamma(t) = (a \cos t, b \sin t)$

$t \in [0, \frac{\pi}{2}]$

$$\int_\gamma F \cdot dx = \int_0^{\frac{\pi}{2}} \left(0, \frac{1}{2} a^2 \cos^2 t \, b \sin t\right) \cdot (-a \sin t, b \cos t) \, dt$$



Note: $F = 0$ on S_1, S_2

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} a^2 b^2 \cos^3 t \sin t \, dt$$

$$= -\int_1^0 \frac{1}{2} a^2 b^2 u^3 \, du = \frac{a^2 b^2}{8}$$

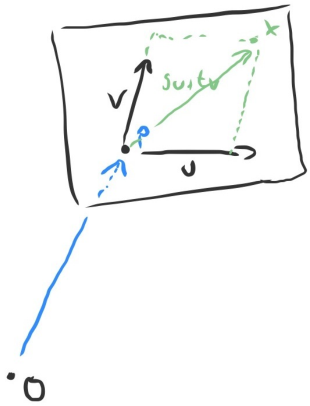
$\gamma'(t) = (-a \sin t, b \cos t)$

$u = \cos t$

$du = -\sin t \, dt$

Planes, normal vectors, and the cross product

Parametric representation of a plane



$$x = p + su + tv$$

$$\Rightarrow \underbrace{\begin{pmatrix} | & | & | \\ p-x & u & v \\ | & | & | \end{pmatrix}}_{A} \begin{pmatrix} 1 \\ s \\ t \end{pmatrix} = 0$$

$\Rightarrow$ A is singular $\Rightarrow$ $\det A = 0$

Review: $\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_1 \det A_{11} - a_2 \det A_{21} + a_3 \det A_{31} - \cdots$

$$= \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \cdot n$$

A_{ij}: matrix obtained from A by removing row i and column j

For the equation of the plane:

$$\underbrace{(p-x)}_{\text{vector lying in the plane}} \cdot n = 0$$

n: normal vector

$$A = \begin{pmatrix} | & u_1 & v_1 \\ p-x & u_2 & v_2 \\ | & u_3 & v_3 \end{pmatrix}$$

$$n_1 = \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} = u_2 v_3 - u_3 v_2$$

$$n_2 = -\det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix} = u_3 v_1 - u_1 v_3$$

$$n_3 = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1$$

we write

$$\boxed{n = u \times v}$$

"cross product"

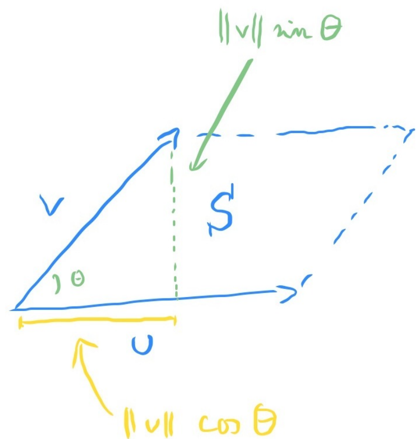
Properties of the cross-product:

(i) $u \times v = -v \times u$

(ii) $\det \begin{pmatrix} | & | & | \\ a & u & v \\ | & | & | \end{pmatrix} = a \cdot (u \times v)$

(iii) $u \times v$ is perpendicular to u and v

(iv) $u \times v = 0$ if u, v l.d.



Area of parallelogram : $S = \|u\| \, \|v\| \sin\theta$

Law of cosines : $u \cdot v = \|u\| \, \|v\| \cos\theta$

$$S^2 = \|u\|^2 \|v\|^2 (1 - \cos^2\theta) = \|u\|^2\|v\|^2 - (u\cdot v)^2 = \|u \times v\|^2 \quad \text{(HW)}$$

$\Rightarrow \|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$

and area of parallelogram spanned by u an v is $\|u \times v\|$

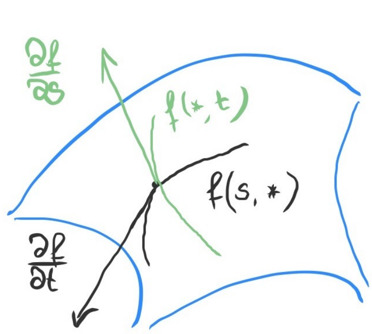
Area and Surface integrals

U domain in $\mathbb{R}^2$, with area

$f \in C(\bar{U}, \mathbb{R}^3)$ $\qquad M = \text{Range } f \subset \mathbb{R}^3$ surface

M is <u>smooth</u> if $f \in C^1(\bar{U}, \mathbb{R}^3)$

and $n = \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} \neq 0$ on $\bar{U}$ $\qquad f(s,t)$



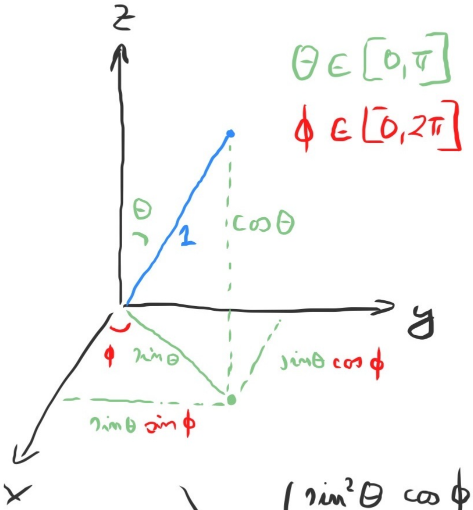
$\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$ are tangent vectors to the plane

$n = \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t}$ normal vector

Area of surface

$$\sigma(M) = \int_U \|n\| \, dS$$

Compare $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$



Example: Surface area of unit sphere

$$f(\Theta, \phi) = (\sin\Theta \cos\phi, \sin\Theta \sin\phi, \cos\Theta)$$

$$\frac{\partial f}{\partial \Theta} = (\cos\Theta \cos\phi, \cos\Theta \sin\phi, -\sin\Theta)$$

$$\frac{\partial f}{\partial \phi} = (-\sin\Theta \sin\phi, \sin\Theta \cos\phi, 0)$$

$$n = \frac{\partial f}{\partial \Theta} \times \frac{\partial f}{\partial \phi} = \begin{pmatrix} 0 - (-\sin\Theta) \sin\Theta \cos\phi \\ -\sin\Theta(-\sin\Theta \sin\phi) - 0 \\ \cos\Theta \cos\phi \sin\Theta \cos\phi - \cos\Theta \sin\phi (-\sin\Theta \sin\phi) \end{pmatrix} = \begin{pmatrix} \sin^2\Theta \cos\phi \\ \sin^2\Theta \sin\phi \\ \cos\Theta \sin\Theta \end{pmatrix}$$

$$\Rightarrow \|n\|^2 = \underbrace{\sin^4\Theta \cos^2\phi + \sin^4\Theta \sin^2\phi}_{\sin^4\Theta} + \cos^2\Theta \sin^2\Theta = \sin^2\Theta (\sin^2\Theta + \cos^2\Theta)$$

$$\Rightarrow \|n\| = \sin\Theta$$

$$\sigma(M) = \int_0^{2\pi} \underbrace{\int_0^{\pi} \sin\Theta \, d\Theta}_{-\cos\Theta \big|_0^{\pi} = 2} \, d\phi = 4\pi$$