Recall:

F conservative $\Leftrightarrow$ $F \cdot dx$ exact

$\Leftrightarrow$ $\int_\gamma F \cdot dx$ depends only on $\gamma(a)$ and $\gamma(b)$

$\Leftrightarrow$ $\int_\gamma F \cdot dx = 0$ if γ is closed

$\Leftrightarrow$ $F = \nabla \phi$

$\Rightarrow$ DF is symmetric
$\overset{*}{\Leftarrow}$

(*) is only true on simply-connected domains ("without holes")
(proved this only for star-shaped domains)

Remark: Special case of Poincaré-Lemma

Examples:

① Let γ be the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$F(x,y) = \left(\frac{y^2}{1+x^2},\ 2y \arctan x\right)$$

Task: compute $\int_\gamma F \cdot dx$

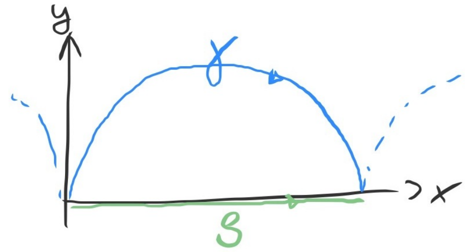
Notice: $\frac{\partial F_1}{\partial y} = \frac{2y}{1+x^2} = \frac{\partial F_2}{\partial x} = \frac{2y}{1+x^2}$

$\Rightarrow$ DF symmetric on $D = \mathbb{R}^2$

$\Rightarrow$ $\int_\gamma F \cdot dx = 0$ as γ is closed.

② $\gamma = (t - \sin t,\ 1 - \cos t)$ (cycloid)

$t \in [0, 2\pi]$



$F = \dfrac{2}{1+x^2+y^2}(x,y)$

$\delta = (t, 0)$

easy to check: F is conservative vector field on $\mathbb{R}^2$

$\delta' = (1,0) = e_1$

Task: compute

$1 + t^2 = u$

$du = 2t\, dt$

$$\int_\gamma F \cdot dx = \int_\delta F \cdot dx = \int_0^{2\pi} F(\delta(t)) \cdot \delta'(t)\, dt$$

$$= \int_0^{2\pi} \frac{2}{1+t^2}(t,0)\cdot(1,0)\, dt = \int_0^{2\pi} \frac{2t}{1+t^2}$$

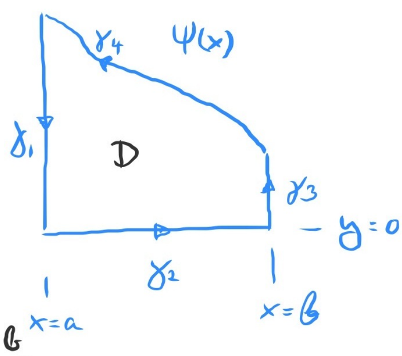
$$= \int_1^{1+(2\pi)^2} \frac{du}{u} = \ln u \Big|_1^{1+(2\pi)^2} = \ln(1 + (2\pi)^2)$$

Green's Theorem (in the plane)

First consider an x-normal domain, WLOG

$$D = \{(x,y) : x \in (a,b),\ 0 \le y \le \psi(x)\}$$

$F = (f, g)$ vector field (C^1)



$$\int_D (-f_y)\, dS = \int_a^b \underbrace{\int_0^{\psi(x)} (-f_y)\, dy}_{= f(x,0) - f(x,\psi(x))}\, dx = \int_a^b f(x,0)\, dx - \int_a^b f(x,\psi(x))\, dx$$

write $\gamma = \partial D$, in anti-clockwise orientation

$$\int_\gamma f\, dx = \int_\gamma (f,0)\cdot d(x,y) = \int_\gamma (f,0)\cdot t\, ds$$

unit tangent vector oriented along the curve

On γ_1, γ_3 tangent vector is perp. to $(f,0)$, so line integrals vanish

$$\int_{\gamma_2} f\, dx = \int_a^b \big(f(x,0), 0\big)\cdot(1,0)\, dx = \int_a^b f(x,0)\, dx$$

$$\int_{\gamma_4} f\, dx = \int_b^a \big(f(x,\psi(x)), 0\big)\cdot(1,\psi'(x))\, dx = -\int_a^b f(x,\psi(x))\, dx$$

Collect pieces:

If D is also y-normal, exchange $x \leftrightarrow y$

$$\int_D (-f_y)\, dS = \int_\gamma f\, dx \qquad \int_D (-g_x)\, dS = -\int_\gamma g\, dy$$

$$\Rightarrow \boxed{\int_D (g_x - f_y)\, dS = \int_\gamma F\cdot d(x,y)}$$

<u>Green's Theorem</u>:

$D \subset \mathbb{R}^2$ bounded, regular(*) domain

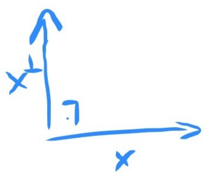
(*) can be decomposed into finitely many bi-normal subdomains

$F \in C^1(\bar{D}, \mathbb{R}^2)$

$$\int_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dS = \int_{\partial D} F\cdot dx$$

where the orientation of the line integral is anti-clockwise.

<u>Note</u>: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, define $x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

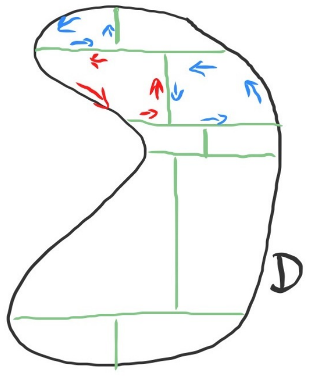


can write

$$\boxed{\int_D \nabla^\perp\cdot F\, dS = \int_{\partial D} F\cdot dx}$$

$$\nabla^\perp\cdot F = \mathrm{curl}_{2D} F \qquad \nabla^\perp = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$$

Proof:



- area integrals over subdomains sum up
- interior pieces of the line integral come up twice with opposite orientation

$\rightarrow$ only boundary pieces remain

Sum up over Green's Theorems for the bi-normal subdomains. $\square$

Examples: ① $F = \frac{1}{2}\begin{pmatrix} -y \\ x \end{pmatrix} = \frac{1}{2}\vec{x}^{\perp}$ $\qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\nabla^{\perp} \cdot F = \frac{1}{2}(-(-1)+1) = 1$$

$$\int_D \nabla^{\perp} \cdot F \, dS = S(D) = \int_{\partial D} F \cdot d(x,y) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

E.g. area of the ellipse with half-axes a, b:

Parametrize ∂D by $\gamma(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$

$$\Rightarrow S(D) = \frac{1}{2}\int_0^{2\pi} \underbrace{\gamma^{\perp}(t) \cdot \gamma'(t)}\, dt \qquad \gamma'(t) = (-a\sin t, b\cos t)$$

$$(-b\sin t, a\cos t) \cdot (-a\sin t, b\cos t) = ab\sin^2 t + ab\cos^2 t$$

$$= \frac{1}{2}\int_0^{2\pi} ab \, dt = \pi a b$$

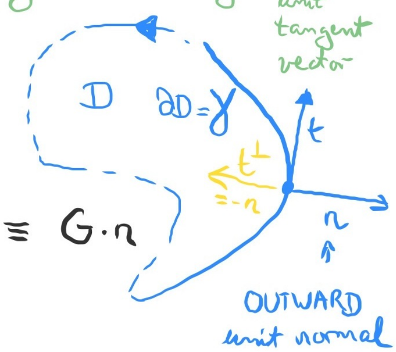
$$\int_\gamma F \cdot dx = \int_\gamma F \cdot t \, ds$$

arc length

unit tangent vector

Remark: $F = G^{\perp}$

$$\int_D \underbrace{\nabla^{\perp} \cdot G^{\perp}}_{= \nabla \cdot G} dS = \int_{\partial D} \underbrace{G^{\perp} \cdot t}_{= (G^{\perp})^{\perp} \cdot t^{\perp}} ds \qquad = -G \cdot t^{\perp} \equiv G \cdot n$$



$$\Rightarrow \quad \boxed{\int_D \underbrace{\nabla \cdot G}_{\operatorname{div} G} \, dS' = \int_{\partial D} G \cdot n \, ds}$$

"Divergence Theorem"

"Fundamental Theorem of Calculus on planar domains"

<u>Compare</u>:

$$\int_a^b f'(x)\,dx = f(x)\Big|_b^a = f(a) - f(b)$$