Normal domains

$\bar{U}$ in $\mathbb{R}^{n-1}$ $\qquad X = (\underbrace{x_1, \ldots, x_{n-1}}_{=: x'}, x_n)$

$$D = \{ x \in \mathbb{R}^n : \phi(x') < x_n < \psi(x'),\ x' \in \bar{U} \}$$

for $\phi, \psi \in C(\bar{U}, \mathbb{R})$

__Remark:__ $$S(D) = \int_D 1\, dS = \int_{\bar{U}} \underbrace{\int_{\phi(x')}^{\psi(x')} dx_n}_{\psi(x') - \phi(x')} \underbrace{d^{n-1}x'}_{dS(x') \text{ or } dS^{n-1}(x')}$$

$$= \int_{\bar{U}} (\psi(x') - \phi(x'))\, d^{n-1}x'$$

__Theorem:__ $f \in C(D)$

$$\Rightarrow \int_D f \underbrace{dx}_{dS(x)} = \int_{\bar{U}} \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n\, d^{n-1}x'$$

To prove, first need a lemma:

__Lemma:__ $D \subset \mathbb{R}^n$ domain with content

$I \subset \mathbb{R}^n$ box s.t. $\bar{D} \subset \operatorname{int} I$

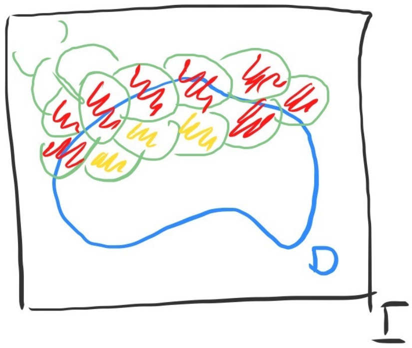
$f \in R(D)$ extend f by zero to I: $f|_{I \setminus \bar{D}} := 0$

$\Rightarrow f \in R(I)$ and $\int_I f\, dx = \int_D f\, dx$ (or: $\int_I f\, dS = \int_D f\, dS$)

Proof of Lemma: $\varepsilon > 0$, look at arbitrary partition of I of

$\lambda(J) < \delta$, δ still to be fixed

$$\sum_j (M_j - m_j)\, S(D_j) = \underbrace{\sum_{\overline{D}_j \subset \operatorname{int}(I\setminus D)} \dots}_{=0} + \underbrace{\boxed{\sum_{\overline{D}_j \cap \partial D \neq \emptyset} \dots}}_{< \frac{\varepsilon}{2}} + \underbrace{\boxed{\sum_{\overline{D}_j \subset \operatorname{int} D} \dots}}_{< \frac{\varepsilon}{2}} < \varepsilon$$



by choosing δ in the definition of content suff. small, $\delta < \delta$ ....

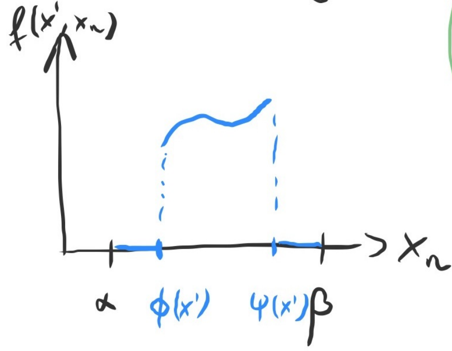
by Riemann integrability criterion on D

$\square$

Proof of Theorem:

$$\int_D f\,dx = \int_I f\,dx = \int_{I'} \underbrace{\int_\alpha^\beta f(x', x_n)\, dx_n}_{\text{1D Riemann integral}}\, dx' = \int_{I'} \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n$$

extend f to I by zero as in Lemma



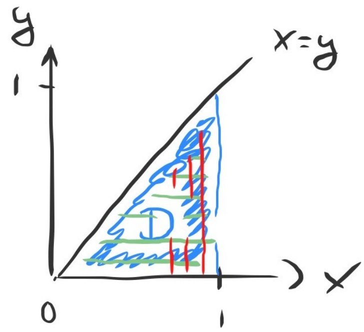
$$= \int_{\overline{U}} \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n\, d^{n-1}x'$$

$\square$

Split $\overline{I'} = \overline{U} \cup \overline{I' \setminus U}$

Examples:

① $\displaystyle\int_0^1 \int_y^1 y e^{x^3}\, dx\, dy$



$$= \int_0^1 \int_0^x y e^{x^3}\, dy\, dx$$

$$= \int_0^1 e^{x^3} \underbrace{\int_0^x y\, dy}_{\frac{1}{2}x^2}\, dx$$

$u = x^3$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$$= \frac{1}{6}\int_0^1 e^u\, du = \frac{1}{6} e^u \Big|_0^1 = \frac{e-1}{6}$$

Change of Variables formula

$U, V \subset \mathbb{R}^n$ domains with content

$\Phi : U \to V$ diffeomorphism : C^1, invertible, inverse is C^1

$f \in R(V)$

$$\boxed{\int_V f\, dx = \int_U f(\Phi(u))\, |\det D\Phi(u)|\, du}$$

Remark: If $\Phi(x) = Ax$, then $D\Phi = A$

det A : area (or content) of I'

Remark: suppose $\Phi(x,y) = \begin{pmatrix} -x \\ y \end{pmatrix}$

$$D\Phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det D\Phi = -1$$

$$\mathrm{F} \overset{\Phi}{\longmapsto} \text{ꟻ}$$

In general, neg. determinants of linear maps indicate that orientation is flipped.

Remark: Compare with integration by substitution: (1D)

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(u))\,\phi'(u)\,du \qquad a < b$$

$$I = [a,b]$$

Case $\phi' < 0$: $\phi(a) > \phi(b)$, and we should write

$$-\int_{\phi(b)}^{\phi(a)} f(x)\,dx = \int_a^b f(\phi(u))\,\phi'(u)\,du$$

so moving the sign to the RHS gives the abstract change-of-variable formula in 1D.

Examples: Polar coordinates in $\mathbb{R}^2$:

$$\Phi(r,\phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$$D\Phi(r,\phi) = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \Rightarrow \det D\Phi = r\cos^2\phi + r\sin^2\phi = r$$

$$\Rightarrow \int_{B(0,R)} f(x)\,dx = \int_0^R \int_0^{2\pi} f(r\cos\phi, r\sin\phi)\,d\phi\, r\,dr$$

$$\Phi : [0,R]\times[0,2\pi) \longrightarrow B(0,R) \quad (\text{diffeo on every } B(0,R)\setminus B(0,\varrho)\ \forall \varrho>0)$$

E.g.

$$\int_{B(0,R)} dx = \int_0^R \underbrace{\int_0^{2\pi} d\theta}_{2\pi}\, r\,dr = 2\pi \tfrac{1}{2}R^2 = \pi R^2$$

area of an ellipse $E = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$

Take $\Phi : B(0,1) \to E \qquad \Phi(u,v) = \begin{pmatrix} au \\ bv \end{pmatrix} \qquad a,b>0$

$$D\Phi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \det\Phi = ab$$

$$\int_E dx = \int_{B(0,1)} |\det\Phi|\, d(u,v) = ab\,\pi$$

Example (gaussian integral):

$$\lim_{R\to\infty} \int_{B(0,R)} e^{-x^2-y^2}\, d(x,y)$$

$$\parallel$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{\mathbb{R}^2} \underbrace{e^{-x^2} e^{-y^2}}_{e^{-x^2-y^2}}\, d(x,y)$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2}\, d\theta\, r\, dr = 2\pi \int_0^{\infty} e^{-r^2}\, r\, dr$$

$u = r^2$

$du = 2r\, dr$

$$= \pi \underbrace{\int_0^{\infty} e^{-u}\, du}_{=1} = \pi \quad \Longrightarrow \quad \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$