Advanced Calculus and Methods of Mathematical Physics

Take-Home Midterm

Tuesday, March 17, 2020

Please use a separate answer sheet for each problem!

1. Consider the power series
$$\sum_{n=0}^{\infty} 2^{n+1}\,(x-1)^n\,.$$
   (a) What is the center and the radius of the ball of convergence?
   (b) Within its ball of convergence, what does the series converge to?
   (c) Does the series converge anywhere on the boundary of the ball of convergence?

   (5+5+5)

2. Let $a>0$, $x_0>0$ and define a sequence by
$$x_{n+1}=F(x_n)\equiv\frac{1}{2}\,x_n+\frac{a}{2\,x_n}\,.$$
   (a) Show that if x_* is a fixed point of F, then $x_*=\sqrt{a}$.
   (b) Show that F is a strict contraction on $[\sqrt{a},\infty)$ and that it maps any interval $[\sqrt{a},b]$, $b>\sqrt{a}$, into itself. Conclude that F has a unique fixed point on this interval.
   (c) What happens if $x_0\in(0,\sqrt{a})$?

   (5+5+5)

3. Let $f\colon\mathbb{R}^2\to\mathbb{R}$ be given by
$$f(x,y)=\ln\frac{1+x^2}{1-y}\,.$$
   (a) Compute the Taylor polynomial (truncated Taylor series) about $(x,y)=(0,0)$ of degree 2.

(b) Give a bound on second order Taylor remainder for $\|(x,y)\| \le \frac{1}{2}$.

(c) On which (open) set does the Taylor series converge? (No need to test convergence on the boundaries.)

(5+5+5)

4. For $A \in \mathrm{Mat}(n \times n)$, define
$$f(A) = (I - A)^{-1}\,,$$
where I is the $n \times n$-identity matrix.

(a) Show that $I - A$ is invertible when $\|A\| < 1$, so that $f(A)$ is well-defined for such matrices.

Hint: A square matrix M is invertible if the homogeneous linear equation $Mx = 0$ has only the trivial solution $x = 0$.

(b) Show that the derivative of f is given by
$$\mathrm{d}f(A)B = (I - A)^{-1} B (I - A)^{-1}\,.$$

(5+5)

5. Let $S = \mathrm{SL}(2, \mathbb{R})$ denote the set of 2×2 matrices with real entries that have determinant 1.[1] Use the implicit function theorem to show that locally, S can be parameterized smoothly by a function of three variables.

Hint: Observe that the determinant condition implies that at least one of the entries of the matrix must be non-zero. Pick that variable and use the IFT to express it locally as a function of the other three. (10)

6. Let $p, q \ge 1$ with
$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

(a) Fix $c > 0$ and maximize
$$f(x,y) = x\,y$$
subject to $x, y \ge 0$ and
$$\frac{1}{p}\,x^p + \frac{1}{q}\,y^q = c\,.$$

(b) Conclude that
$$x\,y \le \frac{1}{p}\,x^p + \frac{1}{q}\,y^q\,.$$

(5+5)

[1] Recall from Linear Algebra that the determinant of a 2×2 matrix is given by
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\,.$$