ACMMP – Mock Midterm Exam Solutions

1(a). Use ratio test:

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{2^{n+2}}{2^{n+1}}\;\frac{(x-1)^{n+1}}{(x-1)^n}\right| = 2\,|x-1|$$

Thus, the series converges absolutely if $2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$,

so the ball of convergence is $B(1, \frac{1}{2})$.

(b) $$\sum_{n=0}^{\infty} 2^{n+1}(x-1)^n = 2\sum_{n=0}^{\infty}\left(2(x-1)\right)^n$$

$$= 2\,\frac{1}{1-2(x-1)} = \frac{2}{3-2x}$$

(c) On the boundary of the ball of convergence, $x-1 = \frac{1}{2}e^{i\theta}$

$$\Rightarrow \sum_{i=0}^{\infty} 2^{n+1}(x-1)^n = 2\sum_{i=0}^{\infty} e^{in\theta}$$

Since $e^{in\theta}$ does not converge to zero, this series does not converge, i.e., the original power series does not converge anywhere on the boundary of the ball of convergence.

2(a). $$x_* = \frac{1}{2}x_* + \frac{a}{2x_*}$$

$$\Rightarrow \frac{1}{2}x_* = \frac{a}{2x_*}$$

$$\Rightarrow x_* = \pm\sqrt{a}$$

On $(0,\infty)$, the fixed point is $x_* = \sqrt{a}$.

(b) $$|F(x)-F(y)| = \left|\frac{1}{2}x + \frac{a}{2x} - \frac{1}{2}y - \frac{a}{2y}\right|$$

$$= \frac{1}{2}\left|x-y + a\underbrace{\left(\frac{1}{x}-\frac{1}{y}\right)}_{=\frac{y-x}{xy}}\right|$$

$$= \frac{1}{2}\underbrace{\left|1-\frac{a}{xy}\right|}_{\in(0,1] \text{ since } x,y\geq\sqrt{a}}|x-y|$$

$$\leq \frac{1}{2}|x-y|$$

$$x - F(x) = x - \frac{1}{2}x - \frac{a}{2x} = \frac{1}{2}x\underbrace{\left(1-\frac{a}{x^2}\right)}_{\geq 0 \text{ for } x\geq\sqrt{a}}$$

$$F(x)-\sqrt{a} = \frac{1}{2}x + \frac{a}{2x} - \sqrt{a}$$

$$= \frac{1}{2x}\left(x^2 - 2\sqrt{a}x + a\right)$$

$$= \frac{1}{2x}(x-\sqrt{a})^2 \geq 0 \quad \text{whenever } x>0. \qquad (*)$$

$\Rightarrow$ F maps any interval $[\sqrt{a}, b]$, $b>\sqrt{a}$, into itself and is a strict contraction on such interval.

Thus, by the contraction mapping theorem, it has a unique fixed point in this interval (namely $x_* = \sqrt{a}$ as we already know) and the sequence of iterates x_n converges to $\sqrt{a}$.

(c) Then $x_1 > \sqrt{a}$, see $(*)$, and the argument proceeds as in (b).

3(a): $$f(x,y) = \ln(1+x^2) - \ln(1-y)$$
$$= x^2 + O(x^3) - \left((-y) - \tfrac{1}{2}(-y)^2 + O(y^3)\right)$$
$$= y + x^2 + \tfrac{1}{2}y^2 + O(\|(x,y)\|^3)$$

Alternatively, write $z = \binom{x}{y}$.

$$f(z) = \underbrace{f(0)}_{=0} + \underbrace{df(0)}_{=(0,1)} z + \tfrac{1}{2} z^T \underbrace{\mathrm{Hess}\, f(0)}_{=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} z + O(\|z\|^3)$$

(b) Here it is easiest to do separate remainder estimates for the single-variable Taylor polynomials in x and y.

$$g(\theta) := \ln(1+\theta)$$

$$g'(\theta) = \frac{1}{1+\theta} \qquad g''(\theta) = -(1+\theta)^{-2} \qquad g'''(\theta) = 2(1+\theta)^{-3}$$

Lagrange remainder : $$R_n(\theta) = \frac{g^{(n+1)}(\xi)}{(n+1)!}\,\theta^{n+1}, \quad |\xi| \le \theta$$

$$\Rightarrow |R_n(\theta)| \le \frac{|g^{(n+1)}(\xi)|}{(n+1)!}\,|\theta|^{n+1}$$

For $\theta = x^2$, $n=1$: $\quad |R_1(\theta)| \leq \frac{1}{2}(1+\xi)^{-2}\theta^2 \quad , \quad |\xi|, |\theta| \leq \frac{1}{4}$

$$\leq \frac{1}{2}\left(1-\frac{1}{4}\right)^{-2}\left(\frac{1}{4}\right)^2 = \frac{1}{18}$$

For $\theta = -y$, $n=2$: $\quad |R_2(\theta)| \leq \frac{1}{6}2(1-\xi)^{-3}\theta^3 \quad , \quad |\xi|, |\theta| \leq \frac{1}{2}$

$$\leq \frac{1}{3}\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{2}\right)^3 = \frac{1}{3}$$

So a reasonable (not quite sharp) bound on the remainder is $\frac{1}{18} + \frac{1}{3} = \frac{7}{18}$.

(c) It converges on $z \in B(0,1)$, due to the radii of convergence of the log series. (Or the slightly larger open box $(-1,1)\times(-1,1)$.)

4(a): Suppose $(I-A)x = 0$

$\Rightarrow \quad x = Ax$

$\Rightarrow \quad \|x\| \leq \|A\|\,\|x\| \qquad$ with $\|A\| < 1$

$\Rightarrow \quad \|x\| = 0$

$\Rightarrow \quad x = 0$

(b) $\quad (I-A)f(A) = I$

$\Rightarrow \quad \delta((I-A)f(A)) = 0$

$\Rightarrow \quad -\delta A\, f(A) + (I-A)\underbrace{\delta f(A)}_{=df(A)\delta A} = 0$

$\Rightarrow \quad df(A)\delta A = (I-A)^{-1}\delta A\, f(A) = (I-A)^{-1}\delta A\,(I-A)^{-1}$.

5. Write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We know that $\det A = ad - bc = 1$

Thus, at least one of the entries is non-zero, say d.

Let $f(a,b,c,d) = ad - bc - 1$.

Then $\frac{\partial f}{\partial a} = d \neq 0$, so that the implicit function theorem asserts the existence of a smooth function $\phi(b,c,d)$ s.t., locally
$$f(\phi(b,c,d), b, c, d) = 0$$
$\Rightarrow S$ is locally parameterized by $A(b,c,d) = \begin{pmatrix} \phi(b,c,d) & b \\ c & d \end{pmatrix}$.

When $d = 0$, $b, c \neq 0$ and a similar argument applies.

6(a) Use Lagrange multipliers:
$$\nabla f = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \nabla g = \begin{pmatrix} x^{p-1} \\ y^{q-1} \end{pmatrix}$$

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{matrix} y = \lambda x^{p-1} \\ x = \lambda y^{q-1} \end{matrix}\right\} \quad \frac{y}{x} = \frac{x^{p-1}}{y^{q-1}} \Rightarrow y^q = x^p$$

This critical point corresponds to a maximum as the constraint set is clearly compact and the minimum is at $x=0$ or $y=0$.

At the critical point,

$$\frac{1}{p}x^p + \frac{1}{q}y^q = \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{=1} x^p = c \qquad \Rightarrow \quad x = c^{\frac{1}{p}}$$

$$= \left(\frac{1}{p}+\frac{1}{q}\right) y^q = c \qquad \Rightarrow \quad y = c^{\frac{1}{q}}$$

$$\Rightarrow f_{max} = c^{\frac{1}{p}} c^{\frac{1}{q}} = c^{\frac{1}{p}+\frac{1}{q}} = c$$

(b) Hence, for arbitrary $x, y \geq 0$

$$f(x,y) \leq f_{max}$$

$$\Rightarrow xy \leq c = \frac{1}{p}x^p + \frac{1}{q}y^q .$$

Note: the inequality

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q \qquad \text{when } \frac{1}{p}+\frac{1}{q} = 1$$

is sometimes called Young's inequality and is an important tool in Analysis.