Advanced Calculus and Methods of Mathematical Physics

Final Exam

May 25, 9:00–11:00

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = (\mathrm{e}^x \cos y, \mathrm{e}^x \sin y)\,.$$

   (a) Show that f is locally invertible at every point $(x,y) \in \mathbb{R}^2$.

   (b) Show that f is not globally one-to-one. Why does this not contradict the inverse function theorem?

(5+5)

2. Let $a, b \in \mathbb{R}^n$ fixed. Consider arbitrary smooth curves $\gamma\colon [0,1] \to \mathbb{R}^n$ that connect a and b, i.e., satisfying $\gamma(0) = a$ and $\gamma(1) = b$. Recall that the length of the curve is given by

$$L = \int_0^1 \|\gamma'(t)\| \,\mathrm{d}t\,.$$

   (a) Show that the derivative of L at a particular curve γ is the linear map acting on an abitrary smooth curve $\phi\colon [0,1] \to \mathbb{R}^n$ satisfying $\phi(0) = \phi(1) = 0$

$$\mathrm{d}L(\gamma)\phi = \int_D \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right) \cdot \phi \,\mathrm{d}t\,.$$

   (b) Conclude from (a) that the length is minimized only if γ is a straight line segment.

(5+5)

3. Minimize $f\colon \mathbb{R}^3 \to \mathbb{R}$,

$$f(x) = x + y + z$$

subject to

$$x^2 + y^2 + z^2 = 1\,,$$
$$x - y - z = 1\,.$$

(10)

4. Find the power series expansion for the function

$$f(x) = \frac{\ln(1+x)}{x}$$

about the point $x = 0$ and determine its radius of convergence. (5)

5. Recall that the Fourier series of a 2π-periodic complex-valued continuous function is given by

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \, \mathrm{e}^{\mathrm{i}kx}$$

where

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} \, f(x) \, \mathrm{d}x \, .$$

Show that for 2π-periodic complex-valued functions f and g,

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} \, g(x) \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} \overline{f_k} \, g_k \, .$$

(You may assume without further discussion that all integrals exist in a suitable sense.) (5)

6. Convert to an integral in polar coordinates and evaluate:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \mathrm{d}y \, \mathrm{d}x \, .$$

(10)

7. Determine whether or not

$$F = (z/\cos^2 x, z, y + \tan x)$$

is conservative on $(-\pi/2, \pi/2) \times \mathbb{R}^2$. If F is conservative, find a potential function for F. (5+5)

8. Let

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

be a vector field defined on $D = \mathbb{R}^2 \setminus \{0\}$.

(a) Show that F is locally conservative.

(b) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

(i) for any simple closed curve γ encircling the origin, (ii) for any simple closed curve γ *not* encircling the origin.

(c) Set $z = x + \mathrm{i}y$. Use the residue theorem to compute

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z$$

for any simple closed curve γ encircling the origin in the complex plane, (ii) for any simple closed curve γ *not* encircling the origin.

(d) State an identification between the real-variable computation from (b) and the complex-variable computation from (c).

(5+5+5+5)

9. Let

$$F(x, y, z) = (y, x\,z, 1)$$

be a vector field in $\mathbb{R}^3$. Let M be the upper hemisphere

$$x^2 + y^2 + z^2 = 1\,, \quad z \geq 0\,.$$

Let $\gamma = \partial M$ be the unit circle in the xy-plane, oriented counter-clockwise when viewed from above.

(a) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

directly.

(b) Compute the same line integral via Stokes' theorem as a surface integral over the capping surface M.

(10+10)