1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = (\mathrm{e}^x \cos y, \mathrm{e}^x \sin y)\,.$$

(a) Show that f is locally invertible at every point $(x,y) \in \mathbb{R}^2$.

(b) Show that f is not globally one-to-one. Why does this not contradict the inverse function theorem?

(5+5)

(a) $Df = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$

$\det Df = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x} > 0$

Thus, Df is invertible for every $(x,y) \in \mathbb{R}^2$, which implies local invertibility of f via the inverse function theorem.

(b) Clearly, f is 2π-periodic w.r.t. y, so it cannot be 1-1 on $\mathbb{R}^2$.

We note that the inverse function theorem only asserts the existence of an open neighborhood about every point on which f is invertible.
It does not make any global claim.

2. Let $a, b \in \mathbb{R}^n$ fixed. Consider arbitrary smooth curves $\gamma\colon [0,1] \to \mathbb{R}^n$ that connect a and b, i.e., satisfying $\gamma(0) = a$ and $\gamma(1) = b$. Recall that the length of the curve is given by

$$L = \int_0^1 \|\gamma'(t)\| \, \mathrm{d}t \,.$$

(a) Show that the derivative of L at a particular curve γ is the linear map acting on an abitrary smooth curve $\phi\colon [0,1] \to \mathbb{R}^n$ satisfying $\phi(0) = \phi(1) = 0$

$$\mathrm{d}L(\gamma)\phi = \int_D \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right) \cdot \phi \, \mathrm{d}t \,. \qquad (*)$$

(b) Conclude from (a) that the length is minimized only if γ is a straight line segment.

(5+5)

(a) Let γ_ε denote a 1-parameter family of curves connecting a and b, and set

$$\delta\gamma = \frac{d}{d\varepsilon}\gamma_\varepsilon \Big|_{\varepsilon=0}$$

Note that $\delta\gamma(0) = 0$ and $\delta\gamma(1) = 0$, as $\gamma_\varepsilon(0) = a$ and $\gamma_\varepsilon(1) = b$ $\forall \varepsilon$.

Then $$\delta L = \int_0^1 \delta \left(\gamma'(t)\cdot\gamma'(t)\right)^{\frac{1}{2}} dt = \int_0^1 \tfrac{1}{2} \|\gamma'(t)\|^{-1} \, 2\gamma'(t)\cdot\delta\gamma'(t) \, dt$$

$$= \underbrace{\frac{\gamma'(t)}{\|\gamma'(t)\|}\cdot\delta\gamma(t)\Big|_{t=0}^{t=1}}_{=0 \ !} - \int_0^1 \frac{d}{dt}\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right)\cdot\delta\gamma(t) \, dt$$

Identifying $\phi \equiv \delta\gamma$, we obtain $(*)$.

(b) L minimal implies that $dL(\gamma)\phi = 0$ for arbitrary smooth ϕ with $\phi(0) = \phi(1) = 1$

$$\Rightarrow \frac{d}{dt}\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right) = 0 \qquad (**)$$

But $\frac{\gamma'}{\|\gamma'\|}$ is a unit vector tangent to γ. $(**)$ says that it does not change along the curve γ, so γ must be a straight line segment.

3. Minimize $f\colon \mathbb{R}^3 \to \mathbb{R}$,

$$f(x) = x + y + z$$

subject to

$$x^2 + y^2 + z^2 = 1\,,$$
$$x - y - z = 1\,.$$

(10)

Use Lagrange multipliers:

$$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$g(x,y,z) \equiv x^2 + y^2 + z^2 - 1 \quad \Rightarrow \quad \nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$h(x,y,z) = x - y - z - 1 \quad \Rightarrow \quad \nabla h = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Necessary condition: $\nabla f + \lambda \nabla g + \mu \nabla h = 0$

$$\Rightarrow \quad \begin{aligned} 1 + \lambda 2x + \mu &= 0 \\ 1 + \lambda 2y - \mu &= 0 \\ 1 + \lambda 2z - \mu &= 0 \end{aligned} \left.\vphantom{\begin{aligned}1\\1\end{aligned}}\right\} \Rightarrow \lambda(y-z) = 0 \Rightarrow \lambda = 0 \text{ or } y = z$$

$\lambda = 0$ is inconsistent with the first equation, so we must have $y = z$.

The two constraints then read

$$x^2 = 1 - 2y^2$$

$$x = 1 + 2y$$

$$\Downarrow \quad x^2 = (1 + 2y)^2 = 1 + 4y + 4y^2$$

$$\Rightarrow \quad 1 - 2y^2 = 1 + 4y + 4y^2$$

$$\Rightarrow \quad 0 = 4y + 6y^2$$

$$\Rightarrow \quad y = 0 \quad \text{or} \quad 2 + 3y = 0 \Leftrightarrow y = -\frac{2}{3}$$

This yields the two candidate points

$$(1, 0, 0) \quad \text{and} \quad \left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$$

where $f(1,0,0) = 1$ and $f\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) = -\frac{5}{3}$

Since the constraint set is compact, f takes its minimum and maximum value on the constraint set, so the two candidate points must correspond to maximum and minimum, respectively.

4. Find the power series expansion for the function

$$f(x) = \frac{\ln(1+x)}{x}$$

about the point $x = 0$ and determine its radius of convergence. (5)

$$\ln(1+x) = \int_0^x \underbrace{\frac{1}{1+t}}_{= 1 - t + t^2 - t^3 + \dots} \, dt$$

$= 1 - t + t^2 - t^3 + \dots$ (geometric series)

The geometric series has radius of convergence 1, and within its radius of convergence, we can integrate term-by-term, so that

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

(this is the standard log series!)

$\Rightarrow$ The singularity at $x = 0$ of f is removable, and

$$f(x) = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n$$

5. Recall that the Fourier series of a 2π-periodic complex-valued continuous function is given by

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \, \mathrm{e}^{\mathrm{i}kx}$$

where

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} f(x) \, \mathrm{d}x \, .$$

Show that for 2π-periodic complex-valued functions f and g,

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} \, g(x) \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} \overline{f_k} \, g_k \, .$$

(You may assume without further discussion that all integrals exist in a suitable sense.) (5)

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} \, g(x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \overline{f_k} \, e^{-ikx} \sum_{j=-\infty}^{\infty} g_j \, e^{ijx} \, dx$$

$$= \frac{1}{2\pi} \sum_{k,j=-\infty}^{\infty} \overline{f_k} \, g_j \underbrace{\int_0^{2\pi} e^{i(j-k)x} \, dx}_{= 2\pi \, \delta_{jk}}$$

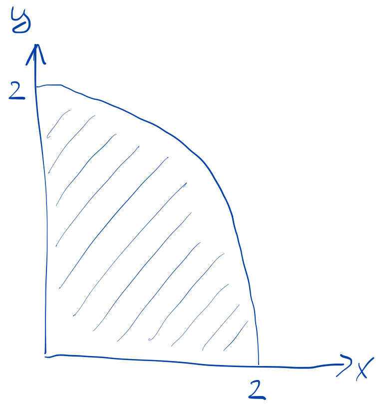
$$= \sum_{k=-\infty}^{\infty} \overline{f_k} \, g_k$$

(This is known as the Parseval identity)

6. Convert to an integral in polar coordinates and evaluate:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \mathrm{d}y\,\mathrm{d}x\,.$$

(10)



$$\int_0^2 \int_0^{\sqrt{4-x^2}} dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^2 r\,dr\,d\varphi$$

$$= \frac{\pi}{2}\,\frac{1}{2}\,2^2 = \pi$$

(area of quarter-circle of radius 2.)

7. Determine whether or not

$$F = (z/\cos^2 x, z, y + \tan x)$$

is conservative on $(-\pi/2, \frac{\pi}{2}) \times \mathbb{R}^2$. If F is conservative, find a potential function for F. (5+5)

$$\nabla \times F = \begin{pmatrix} 1 - 1 \\ \frac{1}{\cos^2 x} - \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} \\ 0 - 0 \end{pmatrix} = 0$$

$\Rightarrow$ F is locally conservative

As the domain is simply connected, this implies that F is globally conservative.

By inspection, $\phi = yz + z\tan x + C$

is a potential function, as $\nabla \phi = F$.

8. Let

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

be a vector field defined on $D = \mathbb{R}^2 \setminus \{0\}$.

(a) Show that F is locally conservative.

(b) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

(i) for any simple closed curve γ encircling the origin, (ii) for any simple closed curve γ *not* encircling the origin.

(c) Set $z = x + \mathrm{i}y$. Use the residue theorem to compute

$$\int_\gamma \frac{1}{z} \,\mathrm{d}z$$

for any simple closed curve γ encircling the origin in the complex plane, (ii) for any simple closed curve γ *not* encircling the origin.

(d) State an identification between the real-variable computation from (b) and the complex-variable computation from (c).

(5+5+5+5)

(a) $$\nabla^\perp \cdot F = -\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{1}{x^2+y^2} + y\frac{-2y}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} + x\frac{-2x}{(x^2+y^2)^2} = 0$$

$\Rightarrow F$ is locally conservative

(b) By (a), the integral is independent of continuous deformations of γ, so:

(i) WLOG, let γ be the unit circle parametrized as

$$\gamma(\phi) = (\cos\phi, \sin\phi), \quad \phi \in [0, 2\pi]$$

$$\Rightarrow \int_\gamma F \cdot ds = \int_0^{2\pi} \underbrace{(-\sin\phi, \cos\phi) \cdot \gamma'(\phi)}_{=1} \, d\phi = 2\pi$$

(ii) the line integral is 0 as γ can be contracted to a point.

(c) $\displaystyle\int_\gamma \frac{1}{z}\,dz = 2\pi i \,\mathrm{Res}\left(\frac{1}{z}, 0\right) = 2\pi i$ if γ encircles the origin

$\displaystyle\int_\gamma \frac{1}{z}\,dz = 0$ otherwise (Cauchy's theorem)

(d) with $z = x + iy$:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = i(u - iv) \qquad \text{with } (u,v) = F$$

$$\Rightarrow \int_\gamma \frac{1}{z}\,dz = \int_\gamma i(u-iv)(dx+i\,dy)$$

$$= \int_\gamma v\,dx - u\,dy + i\underbrace{\int_\gamma u\,dx + v\,dy}_{= i\int_\gamma F\cdot ds}$$

So the first integral must be zero — compare (b) with (c).

This can also be verified by direct computation.

9. Let

$$F(x, y, z) = (y, x\,z, 1)$$

be a vector field in $\mathbb{R}^3$. Let M be the upper hemisphere

$$x^2 + y^2 + z^2 = 1\,, \quad z \geq 0\,.$$

~~restricted to the first octant~~. Let $\gamma = \partial M$ be the unit circle in the xy-plane, oriented counter-clockwise when viewed from above.

(a) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

directly.

(b) Compute the same line integral via Stokes' theorem as a surface integral over the capping surface M.

(10+10)

(a) $\gamma(\phi) = (\cos\phi, \sin\phi, 0)$, $\phi \in [0, 2\pi]$

$\gamma'(\phi) = (-\sin\phi, \cos\phi, 0)$

$$\Rightarrow \int_\gamma F \cdot dx = \int_0^{2\pi} (\sin\phi, 0, 1) \cdot (-\sin\phi, \cos\phi, 0)\, d\phi$$

$$= -\int_0^{2\pi} \underbrace{\sin^2\phi}_{= \frac{1}{2}(1-\cos 2\phi)}\, d\phi \qquad = -\frac{1}{2} 2\pi = -\pi$$

(b) Parametrize M via spherical polar coordinates:

$$f(\phi, \theta) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

$\phi \in [0, 2\pi]$, $\theta \in [0, \frac{\pi}{2}]$

Stokes' theorem:

$$\int_\gamma F \cdot ds = \int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = \int_D (\nabla \times F) \cdot n \, d(\phi, \theta)$$

where $n = \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}$

$$= \begin{pmatrix} \cos\phi \cos\theta \\ \sin\phi \cos\theta \\ -\sin\theta \end{pmatrix} \times \begin{pmatrix} -\sin\phi \sin\theta \\ \cos\phi \sin\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \underbrace{\cos^2\phi \cos\theta \sin\theta + \sin^2\phi \cos\theta \sin\theta}_{= \cos\theta \sin\theta} \end{pmatrix}$$

$$\nabla \times F = \begin{pmatrix} -x \\ 0 \\ z-1 \end{pmatrix}$$

$$\Rightarrow \int_D (\nabla \times F) \cdot n \, d(\phi, \theta) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(\cos\phi \sin^2\theta \sin\phi + (\cos\theta - 1) \cos\theta \sin\theta \right) d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \left(\cos^2\theta \sin\theta - \cos\theta \sin\theta \right) d\theta$$

$\downarrow$ contributes 0 by symmetry (or direct calculation

$$\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta$$

$$= \sin^2\theta \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta = \frac{1}{2}$$

$$= 2\pi \left(-\frac{1}{2}\right) = -\pi$$

This is consistent with (a).